Stability in the inverse Steklov problem on warped product Riemannian manifolds

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Joint work with Niky Kamran (McGill University) and François Nicoleau (Université de Nantes)

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Steklov spectrum

- (M, g): compact connected Riemannian manifold of dimension d with boundary ∂M.
- The Dirichlet-to-Neumann map

$$\Lambda_g \in \mathcal{B}(H^{1/2}(\partial M), H^{-1/2}(\partial M)),$$

is defined by :

$$\Lambda_g \psi = (\partial_\nu u)_{|\partial M},$$

where u solves the Dirichlet problem :

$$\begin{cases} -\triangle_g u = 0, & \text{on } M, \\ u = \psi, & \text{on } \partial M, \end{cases}$$

- The DN map is an elliptic selfadjoint pseudo-differential operator of order 1 on $L^2(\partial M, dS_g)$. Therefore, the DN map has a real and discrete spectrum called the Steklov spectrum

$$0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_k \to \infty.$$

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The inverse Steklov problem

What amount of informations on (M, g) is contained in the Steklov spectrum $(\sigma_k)_{k \geq 0}$?

The inverse Steklov problem

- Weyl's law :

$$\sigma_k = 2\pi\Big(\frac{k}{Vol(\mathbb{B}^{d-1})Vol_g(\partial M)}\Big)^{\frac{1}{d-1}} + O(1).$$

  The dimension d and the volume $Vol_g(M)$ are Steklov spectral invariants

- Heat trace [Polterovich, Sher, (2015)]:

$$\sum_{k=0}^{\infty} e^{-t\sigma_k} = Tr(e^{-t\Lambda_g}) = \sum_{k=0}^{\infty} a_k t^{-n+1+k} + \sum_{l=1}^{\infty} b_l t^l \log t.$$

  For instance, the Steklov spectral invariant a_1 gives the total mean curvature of ∂M.

- In dimension 2, the number and the lengths of the connected components of the boundary ∂M are also Steklov spectral invariants [Girouard, Parnovski, Polterovich, Sher, (2014)].
- Can one hear the shape of a drum? (existence of non isometric manifolds with the same Steklov spectrum by [Gordon, Herbrich, Webb, (2018)]).

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Another motivation: the Calderón conjecture

- Does the DN map Λ_g determine uniquely the metric g modulo pullback of g by diffeomorphisms that preserve the boundary (and conformal scalings in dimension 2)?

- Medical imaging (Electrical Impedance Tomography).

- The Calderon conjecture was solved positively :
  - in dimension 2 for C^∞ metrics,
  - in higher dimensions for real analytic C^w metrics,

  but remains an open problem (in general) in dimensions higher than 3 for C^∞ metrics.

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The geometric model of deformed balls

- We consider warped products (M, g)

$$M = (0,1] \times \mathbb{S}^{d-1}, \quad g = c^4(r)[dr^2 + r^2 g_{\mathbb{S}}].$$

  where
  - $g_{\mathbb{S}}$ is a fixed smooth Riemannian metric on $\mathbb{S}^{d-1}$.
  - c is a positive radial C^m-function with $m \geq 2$ (and $c(0) = 1$).

- The metrics g are generally not regular at $r = 0$, unless $c^{(2k+1)}(0) = 0$ and $g_{\mathbb{S}} = d\Omega^2$ where $d\Omega^2$ is the round metric on $\mathbb{S}^{d-1}$.

- We also work with $x = -\log r \in [0, +\infty)$, so that

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Construction of the Steklov spectrum

- Use the symmetry of the warped product in order to diagonalize the DN map onto the Hilbert basis of the normalized eigenfunctions $\{Y_k\}_{k\geq 0}$ of $-\Delta_{g_{\mathbb{S}}}$, *i.e.*

$$-\Delta_{g_{\mathbb{S}}} Y_k = \mu_k Y_k, \quad \forall k \geq 0,$$

where

$$0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k \to \infty$$

are the ordered eigenvalues of $-\Delta_{g_{\mathbb{S}}}$.

- On each harmonic Y_k, the DN map acts essentially as an operator of multiplication by the Weyl-Titchmarsh function associated to the countable family of Schrödinger operators arising from the separation of variables procedure.

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Separation of variables

- In the coordinate system (x, ω), the Laplace equation $-\Delta_g u = 0$ reads

$$[-\partial_x^2 - \triangle_{g_{\mathbb{S}}} + q_f(x)]v = -\frac{(d-2)^2}{4}v, \qquad v = f^{d-2}u,$$

where

$$q_f(x) = \frac{(f^{d-2})''(x)}{f^{d-2}(x)} - \frac{(d-2)^2}{4} = O(e^{-px}), \quad x \to \infty.$$

- We look for solutions of the form $v = \sum_{k=0}^{\infty} v_k(x) Y_k$. Then $\forall k \geq 0$,

$$-v_k'' + q_f(x)v_k = -\kappa_k^2 v_k, \quad x \in [0, \infty),$$

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$$\kappa_k = \sqrt{\mu_k + \frac{(d-2)^2}{4}}, \quad \forall k \geq 0.$$

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The Weyl-Titchmarsh function

Consider the Schrödinger equation:

$$-v'' + q_f(x)v = zv, \quad x \in [0, \infty).$$

- Let $\{C_0(x,z), S_0(x,z)\}$ be the FSS defined by :

$$C_0(0,z) = 1, \;\; C_0'(0,z) = 0, \;\; S_0(0,z) = 0, \;\; S_0'(0,z) = 1.$$

- For any $z \in \mathbb{C} \setminus \mathbb{R}$, there exists a unique solution $S_\infty(x,z)$ that is L^2 in a neighbourhood of $x = \infty$. Write this function as

$$S_\infty(x,z) = \Delta_q(z)\big(C_0(x,z) - M_q(z)S_0(x,z)\big)\,.$$

- The characteristic function $\Delta_q(z)$ and the Weyl-Titchmarsh function $M_q(z)$ are given by :

$$\begin{aligned} \Delta_q(z) &= W(S_\infty(x,z), S_0(x,z)), \\ M_q(z) &= -\frac{W(C_0(x,z), S_\infty(x,z))}{W(S_0(x,z), S_\infty(x,z))} = \frac{S_\infty'(0,z)}{S_\infty(0,z)}\,. \end{aligned}$$

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A precise formula for the Steklov spectrum

- The DN map can be diagonalized onto the Hilbert basis of harmonics $\{Y_k\}_{k\geq 0}$. If we represent the Dirichlet data as $\psi = \sum_{k\geq 0} \psi_k Y_k$, then the global DN map has the expression

$$\Lambda_g \psi = \sum_{k=0}^{\infty} (\Lambda_g^k \psi_k) Y_k,$$

where

$$\Lambda_g^k \psi_k = \left(\frac{(d-2)f'(0)}{f^3(0)} - \frac{M_q(-\kappa_k^2)}{f^2(0)} \right) \psi_k.$$

- As a consequence, we infer that the Steklov spectrum of (M, g) is given by :

$$\sigma_k = \frac{(d-2)f'(0)}{f^3(0)} - \frac{M_q(-\kappa_k^2)}{f^2(0)}, \quad \forall k \geq 0.$$

Main results

Theorem 1

Assume that $c \in C^\infty([0,1])$. *Then the Steklov spectrum* $(\sigma_k)_{k\geq 0}$ *satisfies for all* $N \in \mathbb{N}$,

$$\sigma_k = \frac{(d-2)f'(0)}{f^3(0)} + \frac{\kappa_k}{f^2(0)} + \sum_{j=0}^{N} \frac{\beta_j(0)}{f^2(0)} \kappa_k^{-j-1} + O(\kappa_k^{-N-2}),$$

as $k \to \infty$, *where*

$$\begin{cases} \beta_0(x) = \frac{1}{2} q_f(x), \\ \beta_{j+1}(x) = \dfrac{1}{2}\beta_j'(x) + \dfrac{1}{2}\displaystyle\sum_{l=0}^{j} \beta_l(x)\beta_{j-l}(x). \end{cases}$$

Some remarks

- Since μ_k and thus κ_k satisfy the usual Weyl law:

$$\kappa_k = 2\pi \left(\frac{k}{Vol(\mathbb{B}^{d-1}) Vol_{g_{\mathbb{S}}}(\mathbb{S})} \right)^{\frac{1}{d-1}} + O(1)\,,$$

  as $k \to \infty$, the Steklov spectrum satisfies the expected Weyl law.

- The coefficients $\beta_j(0),\ j \geq 0$ depend only on the derivatives $q_f^{(l)}(0),\ l = 0, \ldots, j$ up to order j. Hence the values $f(0), f'(0)$ and the Taylor series at 0 of the effective potential q_f give the leading terms of the asymptotics of the Steklov spectrum in inverse powers of κ_k.

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Uniqueness results

Corollary 2

If the warping function c is analytic on $[0,1]$, then the Steklov spectrum of (M,g) determines uniquely the warping function c.

This leads to the question: does the Steklov spectrum determine uniquely the warping function c without assuming analyticity of c ? The answer is yes:

Theorem 3

Let (M,g) and $(M,\tilde{g})$ be Riemannian manifolds of the form

$$g = c^4(r)[dr^2 + r^2 g_{\mathbb{S}}], \quad \tilde{g} = \tilde{c}^4(r)[dr^2 + r^2 g_{\mathbb{S}}]$$

Then $\sigma_k = \tilde{\sigma}_k$, $\forall k \geq 0$ implies that $c = \tilde{c}$.

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A local uniqueness inverse result

We actually have a better local uniqueness result. Precisely

Theorem 4

Let (M, g) and $(M, \tilde{g})$ be Riemannian manifolds with

$$g = c^4(r)[dr^2 + r^2 g_{\mathbb{S}}], \quad \tilde{g} = \tilde{c}^4(r)[dr^2 + r^2 g_{\mathbb{S}}]$$

Then, for a positive constant $a > 0$, the two following assertions are equivalent:

$$\begin{aligned} \sigma_{k^{d-1}} - \tilde{\sigma}_{k^{d-1}} &= O(e^{-2a\kappa_{k^{d-1}}}), \quad k \to \infty. \\ c(r) &= \tilde{c}(r), \quad \forall r \in [e^{-a}, 1]. \end{aligned}$$

Some remarks

- The question of whether the Steklov spectrum determines uniquely the warping function without the additional knowledge of the transversal Riemannian manifold $(\mathbb{S}^{n-1}, g_{\mathbb{S}})$ is false in general (isospectral $(\mathbb{S}, g_{\mathbb{S}})$).

- The above result is a weak form of a stability result: the asymptotic behaviour of the Steklov spectrum allows one to determine the warping function c in a neighbourhood of the boundary $r = 1$.

- In contrast, we shall see that the first Steklov eigenvalues determine in a stable way the warping function c on $[0, 1]$.

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The set of admissible warping functions

Define the set $\mathcal{C}^{m,p}(A)$ of admissible warping functions $c(r)$ where

- A is any positive constant,
- $m \geq 3$,
- $2 \leq p \leq m-1$,

by requiring that $c \in C^m([0,1])$, $c^{(k)}(0) = 0$ for $k = 1, \ldots, p-1$ and

$$\|c\|_{C^m([0,1])} + \|\frac{1}{c}\|_{C^m([0,1])} \leq A\,.$$

If $c \in \mathcal{C}^{m,p}(A)$, the effective potential q_f satisfies the uniform estimates:

$$|q_f^{(k)}(x)| \leq C_A\ e^{-px}\ , \forall x \geq 0,\ \forall k = 0, ..., m-2,$$

where the constant C_A depends only on A.

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Stability for regular deformed balls

Assume that the metric g is regular, *i.e.* we assume $c^{(2k+1)}(0) = 0$, and $g_{\mathbb{S}} = d\Omega^2$:

Theorem 5

Let c, $\tilde{c} \in \mathcal{C}^{m,p}(A)$ where $A > 0$ is fixed. Assume that for some $\epsilon > 0$ one has:

$$\sup_{k \geq 0} |\sigma_k - \tilde{\sigma}_k| \leq \epsilon.$$

Then, there exists a positive constant C_A, depending only on A such that,

$$\|c - \tilde{c}\|_{L^\infty(0,1)} \leq C_A \left(\frac{1}{\log(\frac{1}{\epsilon})} \right)^{p-1}$$

This result is reminiscent of the logarithmic stability estimates obtained by Alessandrini (1988) and Novikov (2011) for Schrödinger operators on bounded domains M in $\mathbb{R}^d$ from the whole DN map.

Stability for singular deformed balls

Theorem 6

Let c, $\tilde{c} \in \mathcal{C}^{m,p}(A)$ where $A > 0$ is fixed. Assume that for some $\epsilon > 0$ one has:

$$\sup_{k \geq 0} |\sigma_k - \tilde{\sigma}_k| \leq \epsilon.$$

Then, there exists $\theta \in (0,1)$ and a positive constant C_A, depending only on A such that,

$$\|c - \tilde{c}\|_{L^\infty(0,1)} \leq C_A \left(\frac{1}{\log(\frac{1}{\epsilon})} \right)^{(p-1)\theta}$$

The formula for the Steklov spectrum

Recall that

$$\sigma_k = \frac{(d-2)f'(0)}{f^3(0)} - \frac{M_q(-\kappa_k^2)}{f^2(0)}, \quad \forall k \geq 0,$$

where

- M_q si the WT function of

$$-v'' + q_f(x)v = -\kappa_k^2 y, \quad x \in [0, \infty),$$

- κ_k^2 are the shifted eigenvalues of $-\Delta_{g_{\mathbb{S}}}$:

$$\kappa_k^2 = \mu_k + \frac{(d-2)^2}{4}, \quad \forall k \geq 0.$$

Some facts concerning the Weyl-Titchmarsh functions M_q

Theorem 7 (Simon, 1999)

There exists a function A on $[0,\infty)$ with the same smoothness as q_f s.t.

$$|A(\alpha) - q_f(\alpha)| \leq Q(\alpha)^2 e^{\alpha Q(\alpha)}, \quad Q(\alpha) = \int_0^\alpha |q_f(s)| ds,$$

such that, if $Re(\kappa) > \frac{1}{2}\|q_f\|_{L^1}$, then

$$M(-\kappa^2) = -\kappa - \int_0^\infty A(\alpha) e^{-2\kappa\alpha} d\alpha.$$

Theorem 8 (Simon, 1999)

The potential q_f on $[0,a]$ is a function of A on $[0,a]$. Explicitly, if q_f and $\tilde{q}_f$ are two potentials, let A and $\tilde{A}$ be their A-functions. Then

$$A(\alpha) = \tilde{A}(\alpha), \ \forall \alpha \in [0,a] \iff q_f(x) = \tilde{q}_f(x), \ \forall x \in [0,a].$$

Some facts concerning the Weyl-Titchmarsh functions M_q

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The asymptotics of the Steklov spectrum

By integration by parts, we get the asymptotics of the Steklov spectrum σ_k, *i.e.* if $c \in C^\infty([0,1])$, then $(\sigma_k)_{k\geq 0}$ satisfies for all $N \in \mathbb{N}$,

$$\sigma_k = \frac{(d-2)f'(0)}{f^3(0)} + \frac{\kappa_k}{f^2(0)} + \sum_{j=0}^{N} \frac{\beta_j(0)}{f^2(0)} \kappa_k^{-j-1} + O(\kappa_k^{-N-2}),$$

as $k \to \infty$, where

$$\begin{cases} \beta_0(x) = \frac{1}{2} q_f(x), \\ \beta_{j+1}(x) = \dfrac{1}{2}\beta_j'(x) + \dfrac{1}{2}\displaystyle\sum_{l=0}^{j} \beta_l(x)\beta_{j-l}(x). \end{cases}$$

Proof of local uniqueness

- Assume that

$$\sigma_{k^{d-1}} - \tilde{\sigma}_{k^{d-1}} = O(e^{-2a\kappa_{k^{d-1}}}), \quad k \to \infty.$$

- The asymptotics on σ_k entail $f(0) = \tilde{f}(0), \quad f'(0) = \tilde{f}'(0)$.
- Hence our assumption reads

$$M(-\kappa^2_{k^{d-1}}) - \tilde{M}(-\kappa^2_{k^{d-1}}) = O(e^{-2a\kappa_{k^{d-1}}}), \quad k \to \infty.$$

- Using the Simon representation, we get

$$\int_0^a \left[A(\alpha) - \tilde{A}(\alpha)\right] e^{-2\kappa_{k^{d-1}}\alpha} d\alpha = O(e^{-2a\kappa_{k^{d-1}}}), \quad k \to \infty.$$

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From discrete to continuous angular momenta

Proposition 9 (Simon, 1999)

Let $f \in L^1(0,a)$. Assume that

$$\int_0^a e^{-xt} f(t)\, dt \;=\; O(e^{-ax}), \quad x \to +\infty.$$

Then, $f = 0$ almost everywhere on $(0,a)$.

Proposition 10 (The discrete case)

Let $f \in L^1(0,a)$. Assume that

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Sketch of proof

- Setting $\nu_k = \frac{1}{c_{d-1}N}\kappa_{(kN)^{d-1}}$, we get from the Weyl law

$$|\nu_k - k| \leq \frac{C}{N} < \frac{1}{4}, \quad \text{for } N \text{ large enough.}$$

- Setting $b = c_{d-1}Na$, and $g(y) = f(\frac{y}{c_{d-1}N})$

$$\int_0^b g(y)e^{-\nu_k y}\, dy = O(e^{-b\nu_k}),\ k \to +\infty,$$

- Define for $z \in \mathbb{C}^+$, $F(z) = e^{bz}\int_0^b g(y)e^{-zy}\, dy$. Then

$$|F(z)| \leq ||g||_1\ e^{bRe(z)}$$

- It follows from $F(\nu_k) = O(1)$ and a theorem of Duffin and Schaeffer that $F(x)$ is bounded for $x > 0$, *i.e.*

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The stability result in the regular case

- Assume that $|\sigma_k - \tilde{\sigma}_k| < \epsilon$ for all $k \geq 0$.
- By letting $k \to +\infty$ in the asymptotics of σ_k, we deduce that $f(0) = \tilde{f}(0)$, and also

$$\left| \frac{(d-2)f'(0)}{f^3(0)} - \frac{(d-2)\tilde{f}'(0)}{\tilde{f}^3(0)} \right| \leq \epsilon.$$

- Hence we obtain :

$$|\tilde{M}(-\kappa_k^2) - M(-\kappa_k^2)| \leq 2f^2(0)\,\epsilon \leq 2A^2\,\epsilon, \quad \forall k \geq 0,$$

where $\kappa_k = k + \frac{d-2}{2}$.

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The Marchenko representation for S_∞

We express the Weyl solution $S_\infty(x,z)$ as :

Lemma 11

Assume that $c \in \mathcal{C}^{m,p}(A)$. Then, there exists a C^{m-1} function $K(x,t)$ for $0 \le x \le t < \infty$, satisfying the properties:

$$\begin{aligned} S_\infty(x,-\kappa^2) &= e^{-\kappa x} + \int_x^{+\infty} K(x,t) e^{-\kappa t}\, dt \ , \ \kappa > 0. \\ K(x,x) &= \frac{1}{2}\int_x^{+\infty} q_f(t)\, dt. \end{aligned}$$

Moreover, there exists a constant $C_A > 0$ depending only on A such that,

$$|\partial_x^k \partial_t^l K(x,t)| \le C_A\ e^{-\frac{p}{2}(x+t)}\ ,\ \forall k,l \le m-1.$$

Another representation formula

For κ sufficiently large, we have:

$$S_\infty(0,-\kappa^2)\tilde{S}_\infty(0,-\kappa^2)\left(M(-\kappa^2)-\tilde{M}(-\kappa^2)\right)$$
$$=\int_0^{+\infty}(q_f(x)-q_{\tilde{f}}(x))S_\infty(x,-\kappa^2)\tilde{S}_\infty(x,-\kappa^2)dx,$$
$$=\int_0^{+\infty}e^{-2\kappa x}B_{q,\tilde{q}}[q_{\tilde{f}}-q_f](x)\ dx,$$

where the operator $B_{q,\tilde{q}}$ is a Volterra type integral operator given by:

$$B_{q,\tilde{q}}h(x)=h(x)+\int_0^x G(x,t)h(t)\ dt,$$

explicit in terms of the Marchenko Kernel $K(x,t)$.

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The operator B

Define the Hilbert spaces

$$\mathcal{H}_\delta = \{q : ||q||^2_{\mathcal{H}_\delta} := \int_0^{+\infty} |q(x)|^2 \, e^{\delta x} \, dx < \infty\}.$$

Then

Proposition 12

Let $c, \tilde{c}$ be warping functions belonging to $\mathcal{C}^{m,p}(A)$. Then, for any $0 < \delta < p$, the operator $B : \mathcal{H}_\delta \to \mathcal{H}_\delta$ is an isomorphism and there exists a constant $C_{A,\delta}$ depending only on A and δ such that

$$\|B\| + \|B^{-1}\| \leq C_{A,\delta}.$$

The moment's approximation problem

- Using this new representation, our main assumption becomes :

$$\left| \int_0^{+\infty} e^{-2\kappa_k x} B[q_{\tilde{f}} - q_f](x)\, dx \right| \leq C_A\, \epsilon, \quad \forall k \geq 0.$$

- Setting $r = e^{-x}$, this can be written as

$$|\int_0^1 r^{\lambda_k}\, h(r)\, dr| \leq C_A\, \epsilon, \quad \forall k \geq 0,$$

where we have set

$$h(r) = r^{-\frac{\delta+1}{2}} B[q_{\tilde{f}} - q_f](-\log r), \qquad \lambda_k = 2k + d - 3 + \frac{\delta+1}{2}.$$

- Note that

$$\|h\|_{L^2(0,1)} = \|B[q_{\tilde{f}} - q_f]\|_{\mathcal{H}_\delta}.$$

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A Hausdorff moment problem with non-integral powers (according to Ang, Gorenflo, Le and Trong)

We shall give an approximation of the L^2-norm of a function f from the approximate knowledge of a finite number of its moments:

$$m_k = \int_0^1 r^{\lambda_k} \, f(r) \, dr \ , \ k \in \mathbb{N}.$$

Thanks to Müntz's Theorem, if

$$\Lambda_\infty = \{0 = \lambda_0 < \lambda_1 < ... < \lambda_n < ...\} \, ,$$

is a sequence of positive real numbers such that

$$\sum_{k=1}^{+\infty} \frac{1}{\lambda_k} = \infty,$$

the system $\{x^{\lambda_0}, \ x^{\lambda_1}, ...\}$ is complete in $L^2([0,1])$.

Müntz polynomials

- Associated to the sequence (λ_k), we define the Müntz polynomials $(L_m(x))$ as $L_0(x) = 1$, and for $m \geq 1$,

$$L_m(x) = \sum_{j=0}^{m} C_{mj} x^{\lambda_j},$$

where we have set

$$C_{mj} = \sqrt{2\lambda_m + 1}\ \frac{\prod_{r=0}^{m-1}(\lambda_j + \lambda_r + 1)}{\prod_{r=0, r\neq j}^{m}(\lambda_j - \lambda_r)}.$$

The family $(L_m(x))$ defines an orthonormal Hilbert basis of $L^2([0,1])$.

- We define the subspace of Müntz "polynomials" of degree λ_n as:

$$\mathcal{M}(\Lambda_n) = \{P : \ P(x) = \sum_{k=0}^{n} a_k\ x^{\lambda_k}\}.$$

Approximation by Müntz polynomials

- Assume that the $(n+1)$ first moments of a function $f \in L^2([0,1])$ are zero up to noise, i.e there exists $\epsilon > 0$ such that

$$m_k = |\int_0^1 f(r)\ r^{\lambda_k}\ dr| \leq \epsilon\ ,\ \forall k = 0, ..., n.$$

- We denote $\pi_n(f)$ the orthogonal projection on the subspace $\mathcal{M}(\Lambda_n)$:

$$\pi_n(f) = \sum_{k=0}^{n} (f, L_k) L_k \implies ||\pi_n(f)||_2^2 \leq \sum_{k=0}^{n} m_k^2 \left(\sum_{p=0}^{k} |C_{kp}| \right)^2 .$$

- We denote the error of approximation of f from $\mathcal{M}(\Lambda_n)$ by :

$$E(f, \Lambda_n)_p := \inf_{P \in \mathcal{M}(\Lambda)} ||f - P||_p.$$

Clearly, one has

$$E(f, \Lambda_n)_2 = ||f - \pi_n(f)||_2 \leq E(f, \Lambda_n)_\infty.$$

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Some estimates on the error

Introduce the index of approximation

$$B(z) := B(z,\Lambda) = \prod_{k=0}^{n} \frac{z-\lambda_k}{z+\lambda_k}, \quad \epsilon_\infty(\Lambda) = \max_{y\geq 0} \left| \frac{B(1+iy)}{1+iy} \right|.$$

Proposition 13

Let $\Lambda : 0 = \lambda_0 < \lambda_1 < \ldots < \lambda_n$ be a finite sequence. Then, for each $f \in C^1([0,1])$,

$$E(f,\Lambda)_\infty \leq 20\ \epsilon_\infty(\Lambda)\ ||f'||_\infty.$$

Corollary 14

Let $\Lambda_n^* : 0 < \lambda_1 < \ldots < \lambda_n$ be a finite sequence. For $0 \leq k \leq p-1$, we set: $\Lambda_n^{(k)} :\ \lambda_1^{(k)} = \lambda_1 - k,\ \ldots,\ \lambda_n^{(k)} = \lambda_n - k$. Then for each $f \in C^p([0,1])$ such that $f^{(k)}(0) = 0$ for all $k = 0, \ldots, p-1$, one has:

$$E(f,\Lambda_n^*)_\infty \leq 40^p \prod_{k=0}^{p-1} \epsilon_\infty(\Lambda_n^{(k)})\ \ ||f^{(p)}||_\infty.$$

Some estimates on the error

Introduce the index of approximation

$$B(z) := B(z,\Lambda) = \prod_{k=0}^{n} \frac{z-\lambda_k}{z+\lambda_k}, \quad \epsilon_\infty(\Lambda) = \max_{y\geq 0} \left| \frac{B(1+iy)}{1+iy} \right|.$$

Proposition 13

Let $\Lambda : 0 = \lambda_0 < \lambda_1 < \ldots < \lambda_n$ be a finite sequence. Then, for each $f \in C^1([0,1])$,

$$E(f,\Lambda)_\infty \leq 20\ \epsilon_\infty(\Lambda)\ ||f'||_\infty.$$

Corollary 14

Let $\Lambda_n^* : 0 < \lambda_1 < \ldots < \lambda_n$ be a finite sequence. For $0 \leq k \leq p-1$, we set: $\Lambda_n^{(k)} :\ \lambda_1^{(k)} = \lambda_1 - k,\ \ldots,\ \lambda_n^{(k)} = \lambda_n - k$. Then for each $f \in C^p([0,1])$ such that $f^{(k)}(0) = 0$ for all $k = 0, \ldots, p-1$, one has:

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Application

Applying this to our moment conditions

$$|\int_0^1 r^{\lambda_k}\ h(r)\ dr| \leq C_A\ \epsilon,\ \forall k \geq 0,$$

we obtain after some work:

$$\begin{aligned}
\|h\|_2^2 &= \|\pi_n(h)\|_2^2 + \|h - \pi_n(h)\|_2^2, \\
&\leq \epsilon^2 \sum_{k=0}^{n} \left(\sum_{p=0}^{k} |C_{kp}| \right)^2 + \left(40^p \prod_{k=0}^{p-1} \epsilon_\infty(\Lambda_n^{(k)})\ \ \|h^{(p-1)}\|_\infty \right)^2, \\
&\leq B^2 \epsilon^2\ g(n)^2 + C_A \left(\frac{1}{n}\right)^{p-1},
\end{aligned}$$

for some constant B, where (for some constant M)

$$g(t) = \frac{3}{2} \frac{1}{\sqrt{\left(\frac{9M}{2}\right)^2 - 1}} \sqrt{2t+1} \left(\frac{9M}{2}\right)^{t+1}.$$

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- Set $n(\epsilon) := E\ (g^{-1}(\frac{1}{\sqrt{\epsilon}}))$. Clearly, since g is an increasing function, one has $g(n(\epsilon)) \leq \frac{1}{\sqrt{\epsilon}}$. Hence we get :

$$||\pi_{n(\epsilon)} h||_2^2 \leq B^2 \epsilon.$$

- Moreover, a straightforward calculation shows that

$$n(\epsilon) \sim C \log(\frac{1}{\epsilon}),\ \epsilon \to 0,$$

- As a conclusion, we have obtained for ϵ small enough:

$$\|h\|_{L^2} = ||B[q_{\tilde{f}} - q_f]||_{\mathcal{H}_\delta} \leq C_A \left(\frac{1}{\log(\frac{1}{\epsilon})} \right)^{p-1}.$$

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Some perspectives and open questions

- The reconstruction problem of the conformal factor from the Steklov spectrum is reachable (work in progress).
- Can uniqueness and better (Hölder, Lipschitz) stability estimates be obtained using other spectral data such as Regge poles ? (work in progress).
- The methods used in this work are entirely based on 1-d techniques adapted to a radial conformal factor. Can the conformal factor be perturbed in transversal directions?
- Can doubly warped products, or more general separable structures (Stäckel or Painlevé systems) be considered? This could allow for special metrics on the angular part.