# System of radiative transfer equations for coupled surface and body waves

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November 17, 2021

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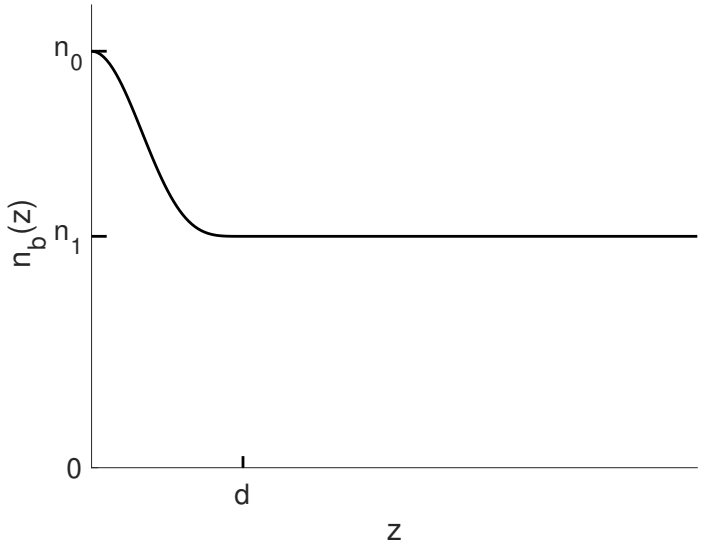
# Motivations

- Radiative transfer theory describes the propagation of wave energy in random media in the following regime:
  - Medium with weak fluctuations at scale $\sim$ wavelength
  - Propagation distance $\gg$ wavelength.
- Radiative transfer in open space (proposed phenomenologically by Chandrasekhar) is well established.
- Boundary effects are intensively studied.
- Main objective: description of energy transport in seismic coda, in particular, transfer of energy between surface waves and body waves.
- Idea: Derive a system of radiative transfer equations for coupled "surface" and "body" waves in a scalar approximation [Margerin 19].
- Framework: A two-dimensional scalar model involving a thick waveguide or a half-space with a thin surface layer and random heterogeneities.

# Propagation in a half-space containing a non-scattering thin layer

$$\frac{\mathrm{n_b^2}(z)}{c_o^2}\partial_t^2 u - \Delta u = \delta(x) f(z;t), \quad (x,z;t) \in \mathbb{R} \times (0,+\infty) \times \mathbb{R},$$

with $u(x, z = 0; t) = 0$, $\Delta = \partial_x^2 + \partial_z^2$ and $\mathrm{n_b}$ smooth, non-increasing, of the form




Take Fourier transform: $\hat{u}(x,z;\omega) = \int_{\mathbb{R}} u(x,z;t)\exp(i\omega t)dt$

$$\Delta \hat{u} + k^2 \mathrm{n_b^2}(z)\hat{u} = -\delta(x)\hat{f}(z;\omega), \quad (x,z) \in \mathbb{R} \times (0,+\infty),$$

with $k = \omega / c_o$.

**Spectral problem** associated to the one-dimensional Schrödinger operator $(\partial_z^2 + k^2 \mathrm{n}_\mathrm{b}^2(z))\phi(z) = \gamma\phi(z)$ with Dirichlet boundary condition at $z = 0$:

- The spectrum is of the form $(-\infty, n_1^2 k^2) \cup \{\beta_{N-1}^2, \ldots, \beta_0^2\}$.
- The $N$ modal wavenumbers $\beta_j$ are in $(n_1 k, n_0 k)$.
- The functions $\phi_j$, $j = 0, \ldots, N-1$, are the modes corresponding to the discrete spectrum. They decay exponentially in $z$ for $z > d$.
- The functions $\phi_\gamma$, $\gamma \in (-\infty, n_1^2 k^2)$, are the modes corresponding to the continuous spectrum. They are oscillatory and bounded at infinity.
- The set of modes is complete in $L^2(0, +\infty)$.
  Any function $v \in L^2(0, +\infty)$ can be expanded on this complete set:

$$v(z) = \sum_{j=0}^{N-1} v_j \phi_j(z) + \int_{-\infty}^{n_1^2 k^2} v_\gamma \phi_\gamma(z) d\gamma,$$

with $v_j = (\phi_j, v)_{L^2}$ and $v_\gamma = (\phi_\gamma, v)_{L^2}$.

- We note that $\phi_\gamma$ does not belong to $L^2(0,+\infty)$, but $(\phi_\gamma, v)_{L^2}$ can be defined for any test function $v \in L^2(0,+\infty)$ as

$$(\phi_\gamma, v)_{L^2} = \lim_{D\to+\infty} \int_0^D \phi_\gamma(z) v(z) dz,$$

  where the limit holds (as a function in $\gamma$) in $L^2(-\infty, n_1^2 k^2)$.
- We have an isometry from $L^2(0,+\infty)$ onto $\mathbb{C}^N \times L^2(-\infty, n_1^2 k^2)$ with

$$(v,v)_{L^2} = \sum_{j=0}^{N-1} \left| (\phi_j, v)_{L^2} \right|^2 + \int_{-\infty}^{n_1^2 k^2} \left| (\phi_\gamma, v)_{L^2} \right|^2 d\gamma.$$

Cf. Magnanini and Santosa 00 (using the Levitan-Levinson transform method).

- The solution of the Helmholtz equation can be expanded as:

$$\hat{u}(x,z) = \sum_{j=0}^{N-1} \hat{u}_j(x)\phi_j(z) + \int_{-\infty}^{n_1^2 k^2} \hat{u}_\gamma(x)\phi_\gamma(z)d\gamma.$$

- The complex mode amplitudes satisfy for $x \in (0,+\infty)$:

$$\partial_x^2 \hat{u}_j + \beta_j^2 \hat{u}_j = 0, \quad j = 0,\ldots,N-1,$$
$$\partial_x^2 \hat{u}_\gamma + \gamma \hat{u}_\gamma = 0, \quad \gamma \in (-\infty, n_1^2 k^2).$$

- Therefore we have for $x \in (0,+\infty)$:

$$\hat{u}(x,z) = \sum_{j=0}^{N-1} \frac{a_{j,\mathrm{s}}}{\sqrt{\beta_j}} e^{i\beta_j x}\phi_j(z) + \int_0^{n_1^2 k^2} \frac{a_{\gamma,\mathrm{s}}}{\gamma^{1/4}} e^{i\sqrt{\gamma}x}\phi_\gamma(z)d\gamma$$
$$+ \int_{-\infty}^{0} \frac{a_{\gamma,\mathrm{s}}}{|\gamma|^{1/4}} e^{-\sqrt{|\gamma|}x}\phi_\gamma(z)d\gamma,$$

where $a_{j,\mathrm{s}}$, $a_{\gamma,\mathrm{s}}$ are constant and determined by the source.

- The modes for $0 \le j \le N-1$ are guided ("surface modes"), the modes for $\gamma \in (0, n_1^2 k^2)$ are radiating ("body modes"), the modes for $\gamma \in (-\infty, 0)$ are evanescent.

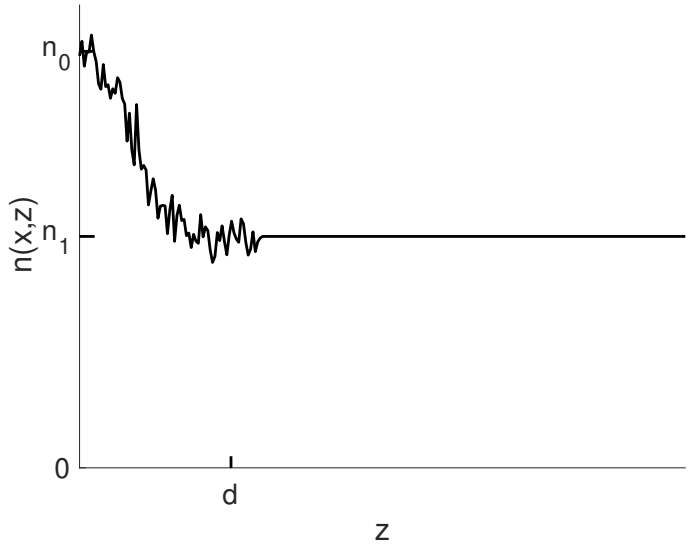
# Propagation in a half-space containing a scattering thin layer

$$\frac{n^2(x,z)}{c_o^2}\partial_t^2 u - \Delta u = \delta(x) f(z;t), \quad (x,z;t) \in \mathbb{R} \times (0,+\infty) \times \mathbb{R}.$$

We consider that the thin layer is scattering

$$n^2(x,z) = \mathrm{n}_\mathrm{b}^2(z) + \varepsilon \nu(x,z), \quad z \in (0,+\infty),$$

where $\nu$ is a zero-mean, bounded, random process, stationary in $x$, with integrable in $x$ covariance, compactly supported in $z$:



- We still have

$$\hat{u}(x,z) = \sum_{j=0}^{N-1} \hat{u}_j(x)\phi_j(z) + \int_{-\infty}^{n_1^2 k^2} \hat{u}_\gamma(x)\phi_\gamma(z)d\gamma,$$

but the $\hat{u}_j(x)$, $\hat{u}_\gamma(x)$ satisfy coupled equations:

$$\partial_x^2 \hat{u}_j + \beta_j^2 \hat{u}_j = -\varepsilon k^2 \sum_{l=0}^{N-1} C_{j,l}(x)\hat{u}_l - \varepsilon k^2 \int_{-\infty}^{n_1^2 k^2} C_{j,\gamma'}(x)\hat{u}_{\gamma'} d\gamma',$$

for $j = 0, \ldots, N-1$,

$$\partial_x^2 \hat{u}_\gamma + \gamma \hat{u}_\gamma = -\varepsilon k^2 \sum_{l=0}^{N-1} C_{\gamma,l}(x)\hat{u}_l - \varepsilon k^2 \int_{-\infty}^{n_1^2 k^2} C_{\gamma,\gamma'}(x)\hat{u}_{\gamma'} d\gamma',$$

for $\gamma \in (-\infty, n_1^2 k^2)$, where

$$C_{j,l}(x) = (\phi_j, \phi_l \nu(x,\cdot))_{L^2}, \qquad C_{j,\gamma'}(x) = (\phi_j, \phi_{\gamma'} \nu(x,\cdot))_{L^2},$$
$$C_{\gamma,l}(x) = (\phi_\gamma, \phi_l \nu(x,\cdot))_{L^2}, \qquad C_{\gamma,\gamma'}(x) = (\phi_\gamma, \phi_{\gamma'} \nu(x,\cdot))_{L^2}.$$

- Consider large propagation time and distance $\sim \varepsilon^{-2}$.

Quantity of interest (Wigner transform of the normal derivative of the field at the surface):

$$
\begin{aligned}
W^{\mathrm{s}}(x,\kappa;t,\omega) &= \lim_{\varepsilon\to 0}\frac{1}{2\pi}\iint d\omega' dx' \exp(-i\omega' t - i\kappa x') \\
&\quad \times \mathbb{E}\Big[\partial_z \hat{u}(\frac{x}{\varepsilon^2}+\frac{x'}{2},0;\omega+\frac{\varepsilon^2}{2}\omega')\partial_z \overline{\hat{u}}(\frac{x}{\varepsilon^2}-\frac{x'}{2},0;\omega-\frac{\varepsilon^2}{2}\omega')\Big] \\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\iint dt' dx' \exp(i\omega t' - i\kappa x') \\
&\quad \times \mathbb{E}\Big[\partial_z u(\frac{x}{\varepsilon^2}+\frac{x'}{2},0;\frac{t}{\varepsilon^2}+\frac{t'}{2})\partial_z u(\frac{x}{\varepsilon^2}-\frac{x'}{2},0;\frac{t}{\varepsilon^2}-\frac{t'}{2})\Big].
\end{aligned}
$$

It represents the energy density at time $t/\varepsilon^2$ and frequency $\omega$ that arrives at $x/\varepsilon^2$ with the angle determined by the longitudinal wavenumber $\kappa$.

- The Wigner transform $W^{\mathrm{s}}$ is of the form:

$$W^{\mathrm{s}}(x,\kappa;t,\omega) = \sum_{j=0}^{N(\omega)-1} \frac{\partial_z \phi_j(0;\omega)^2}{\beta_j(\omega)} W_j(x;t,\omega)\delta(\kappa-\beta_j(\omega)),$$

where the $W_j(x;t,\omega)$'s satisfy

$$\partial_x W_j + \frac{1}{v_j(\omega)}\partial_t W_j = \sum_{l=0,l\neq j}^{N(\omega)-1} \Gamma_{jl}(\omega) W_l - \Lambda_j(\omega) W_j.$$

- $\beta_j$ is the phase velocity of the $j$-th surface mode,
- $v_j$ is the group velocity of the $j$-th surface mode,
- $\Gamma_{jl}$ is the scattering coefficient (energy coming from the $l$-th surface mode to the $j$-th surface mode),
- $\Lambda_j$ is the extinction coefficient, that takes into account leakage towards the body modes and scattering to other surface modes.

The scattering and extinction coefficients depend on the two-point statistics of the fluctuations of the random medium.

- The $W_j(x;t,\omega)$'s satisfy

$$\partial_x W_j + \frac{1}{v_j(\omega)}\partial_t W_j = \sum_{l=0,l\neq j}^{N(\omega)-1} \Gamma_{jl}(\omega) W_l - \Lambda_j(\omega) W_j,$$

with $v_j(\omega) = 1/\partial_\omega \beta_j(\omega)$,

$$\begin{aligned}
\Gamma_{jl}(\omega) =& \frac{k^4(\omega)}{2\beta_j\beta_l(\omega)} \int_0^\infty \mathcal{R}_{jl}(x;\omega) \cos\big((\beta_l(\omega)-\beta_j(\omega))x\big)\,dx,\\
\Lambda_j(\omega) =& \sum_{l=0,l\neq j}^{N(\omega)-1} \Gamma_{jl}(\omega)\\
&+ \int_0^{n_1^2 k^2(\omega)} \frac{k^4(\omega)}{2\sqrt{\gamma}\beta_j(\omega)} \int_0^\infty \mathcal{R}_{j\gamma}(x;\omega)\cos\big((\sqrt{\gamma}-\beta_j(\omega))x\big)\,dx d\gamma,\\
\mathcal{R}_{jl}(x;\omega) =& \int_0^\infty \int_0^\infty \phi_j\phi_l(z;\omega)\mathbb{E}[\nu(0,z)\nu(x,z')]\phi_j\phi_l(z';\omega)\,dz dz'.
\end{aligned}$$

$\rightarrow$ Energy is lost to the surface modes ($\Lambda_j(\omega) > \sum_{l=0,l\neq j}^{N(\omega)-1} \Gamma_{jl}(\omega)$).

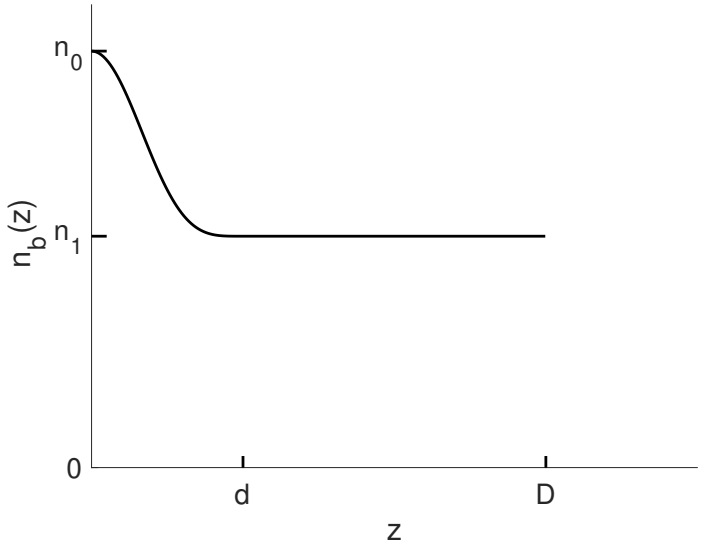
# Towards the associated inverse problem

- From the observation of the seismic coda (Wigner tranform), one could extract (in a frequency band):
  - the phase and group velocities $\beta_j(\omega)$ and $\partial_\omega \beta_j(\omega)$ for the surface modes $j \leq N(\omega)$ (typical for standard surface wave tomography),
  - the normal derivatives $\partial_z \phi_j(z=0,\omega)^2$,
  - the scattering coefficients $\Gamma_{jl}(\omega)$.
    With a delta-correlation approximation for the covariance of $\nu$, $\Gamma_{jl}(\omega)$ is proportional to $\int_0^\infty \phi_j(z;\omega)^2 \phi_l(z,\omega)^2 dz$.
- Can we estimate the background index of refraction $\mathrm{n_b}$ ?

# Propagation in a thick layer containing a thin layer

$$\frac{\mathrm{n_b^2}(z)}{c_o^2}\partial_t^2 u - \Delta u = \delta(x) f(z;t), \quad (x,z;t) \in \mathbb{R} \times (0,D) \times \mathbb{R},$$

with $u(x, z = 0; t) = 0$, $\partial_z u(x, z = D; t) = 0$, and $\mathrm{n_b}$ of the form



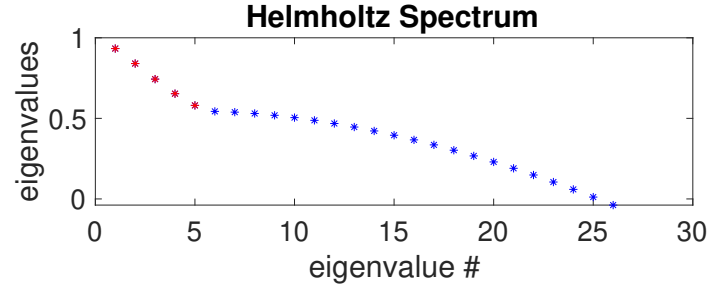

Take Fourier transform: $\hat{u}(x,z;\omega) = \int_{\mathbb{R}} u(x,z;t)\exp(i\omega t)dt$

$$\Delta \hat{u} + k^2 \mathrm{n_b^2}(z)\hat{u} = -\delta(x)\hat{f}(z;\omega), \quad (x,z) \in \mathbb{R} \times (0,D),$$

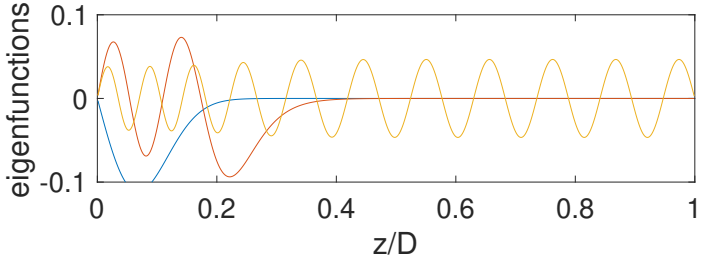
with $k = \omega/c_o$.

**Spectral problem** associated to the one-dimensional Schrödinger operator $(\partial_z^2 + k^2 \mathrm{n}_\mathrm{b}^2(z))\phi(z) = \gamma\phi(z)$ in $L^2(0, D)$ with Dirichet boundary condition at $z = 0$ and Dirichlet or Neumann boundary condition at $z = D$:

- The spectrum is discrete. The eigenvalues are of the form $\gamma_{j,D}$ with $\cdots < \gamma_{j+1,D} < \gamma_{j,D} < \cdots < \gamma_{0,D} < n_0^2 k^2$.
- We denote $N_D$ such that $\gamma_{N_D,D} \leq n_1^2 k^2 < \gamma_{N_D-1,D}$, we denote $M_D$ such that $\gamma_{M_D,D} \leq 0 < \gamma_{M_D-1,D}$.
- For $j \leq M_D$ we write $\beta_{j,D}(\omega) = \sqrt{\gamma_{j,D}}$.
- The eigenfunctions $\phi_{j,D} \in L^2(0, D)$ for all $j$. They are exponentially decaying in $(d, D)$ for $j < N_D$ and oscillatory for $j \geq N_D$.




Here $d = 0.2D$, $N_D = 5$, $M_D = 26$.




$\phi_{j,D}(z)$ for $j = 1, 4, 20$.

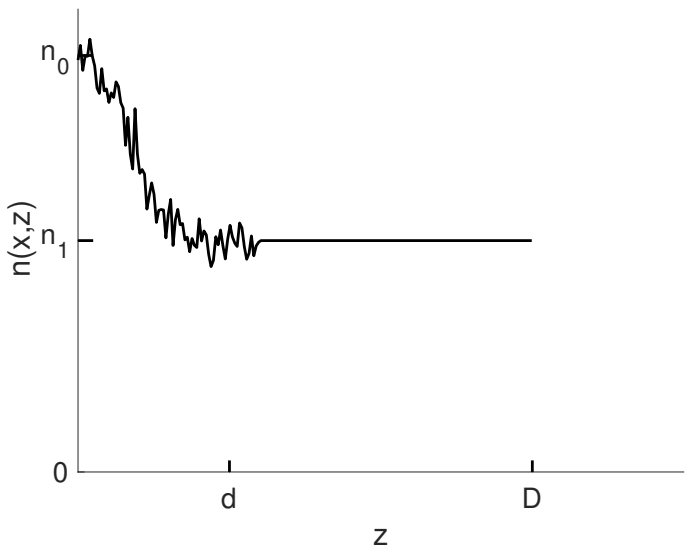
- The set of eigenfunctions is complete in $L^2(0, D)$.

# Propagation in a thick layer containing a thin scattering layer

$$\frac{n^2(x,z)}{c_o^2}\partial_t^2 u - \Delta u = \delta(x) f(z;t), \quad (x,z;t) \in \mathbb{R} \times (0,D) \times \mathbb{R},$$

with $u(x, z = 0; t) = 0$, $\partial_z u(x, z = D; t) = 0$, and $n$ of the form

$$n^2(x,z) = \mathrm{n}_\mathrm{b}^2(z) + \varepsilon \nu(x,z)$$




- We can write a RTE for the truncated problem with a fixed $D$ as $\varepsilon \to 0$.

In this truncated problem there are only discrete modes.

- Then we consider $kD \to +\infty$.

- In the regime $\varepsilon \to 0$, the Wigner transform $W^{\rm s}$ is of the form:

$$W_D^{\rm s}(x,\kappa;t,\omega) = \sum_{j=0}^{M_D(\omega)-1} \frac{\partial_z \phi_{j,D}(0;\omega)^2}{\beta_{j,D}(\omega)} W_{j,D}(x;t,\omega)\delta\big(\kappa - \beta_{j,D}(\omega)\big),$$

where the $W_{j,D}(x;t,\omega)$'s satisfy

$$\partial_x W_{j,D} + \frac{1}{v_{j,D}(\omega)}\partial_t W_{j,D} = \sum_{l=0,l\neq j}^{M_D(\omega)-1} \Gamma_{jl,D}(\omega) W_{l,D} - \Lambda_{j,D}(\omega) W_{j,D}.$$

$\to$ No energy is lost:

$$\partial_x W_{j,D} + \frac{1}{v_{j,D}(\omega)}\partial_t W_{j,D} = \sum_{l=0,l\neq j}^{M_D(\omega)-1} \Gamma_{jl,D}(\omega)\big(W_{l,D} - W_{j,D}\big).$$

- In the regime $\varepsilon \to 0$, the Wigner transform $W^{\rm s}$ is of the form:

$$W_D^{\rm s}(x,\kappa;t,\omega) = \sum_{j=0}^{M_D(\omega)-1} \frac{\partial_z \phi_{j,D}(0;\omega)^2}{\beta_{j,D}(\omega)} W_{j,D}(x;t,\omega)\delta(\kappa - \beta_{j,D}(\omega)),$$

where the $W_{j,D}(x;t,\omega)$'s satisfy

$$\partial_x W_{j,D} + \frac{1}{v_{j,D}(\omega)}\partial_t W_{j,D} = \sum_{l=0,l\neq j}^{M_D(\omega)-1} \Gamma_{jl,D}(\omega) W_{l,D} - \Lambda_{j,D}(\omega) W_{j,D},$$

with $v_{j,D}(\omega) = 1/\partial_\omega \beta_{j,D}(\omega)$,

$$\Gamma_{jl,D}(\omega) = \frac{k^4(\omega)}{2\beta_{j,D}\beta_{l,D}(\omega)} \int_0^\infty \mathcal{R}_{jl,D}(x;\omega)\cos\left((\beta_{l,D}(\omega) - \beta_{j,D}(\omega))x\right)dx,$$

$$\mathcal{R}_{jl,D}(x;\omega) = \int_0^D \int_0^D \phi_{j,D}\phi_{l,D}(z;\omega)\mathbb{E}[\nu(0,z)\nu(x,z')]\phi_{j,D}\phi_{l,D}(z';\omega)dzdz',$$

$$\Lambda_{j,D}(\omega) = \sum_{l=0,l\neq j}^{M_D(\omega)-1} \Gamma_{jl,D}(\omega).$$

• Asymptotic analysis $kD \to +\infty$ [Codington-Levinson 84].
For any $\gamma \in \mathbb{R}$ we denote by $\psi_\gamma$ the unique solution of

$$(\partial_z^2 + k^2 \mathrm{n}_\mathrm{b}^2(z))\psi_\gamma(z) = \gamma\psi_\gamma(z), \quad z \in (0, +\infty),$$

starting from $\psi_\gamma(z = 0) = 0$ and $\partial_z\psi_\gamma(z = 0) = 1$.
$\to$ $\gamma_{j,D}$ satisfies an algebraic condition (boundary condition at $z = D$).
$\to$ $\phi_{j,D}(z) = \sqrt{r_{j,D}}\psi_{\gamma_{j,D}}(z)$, $r_{j,D}^{-1} = \int_0^D \psi_{\gamma_{j,D}}(z)^2 dz$.
$\to$ If $v \in L^2(0, +\infty)$, we have [Sturm-Liouville theory]

$$v(z) = \sum_{j=0}^{\infty} v_{j,D}\phi_{j,D}(z) \text{ in } (0, D), \quad \text{where} \quad v_{j,D} = \int_0^D v(z)\phi_{j,D}(z)dz.$$

- Asymptotic analysis $kD \to +\infty$ [Codington-Levinson 84].

If $v \in L^2(0,+\infty)$, then we have

$$v(z) = \int_{\mathbb{R}} V_D(\gamma)\psi_\gamma(z)\rho_D(d\gamma) \quad \text{in } (0,D),$$

where

$$V_D(\gamma) = \int_0^D v(z)\psi_\gamma(z)dz \quad \text{and} \quad \rho_D(d\gamma) = \sum_{j=0}^{\infty} r_{j,D}\delta_{\gamma_{j,D}}(d\gamma).$$

Result: $V_D \to V$ and $\rho_D \to \rho$ as $D \to +\infty$ in the proper topologies, where

$$V(\gamma) = \int_0^\infty v(z)\psi_\gamma(z)dz$$

$$\rho(d\gamma) = \sum_{j=0}^{N-1} r_j\delta_{\gamma_j}(d\gamma) + r_\gamma \mathbf{1}_{(-\infty,n_1^2k^2)}(\gamma)d\gamma$$

with

$$r_j := \frac{2\sqrt{\gamma_j - n_1^2k^2}}{2\sqrt{\gamma_j - n_1^2k^2}\int_0^d \psi_{\gamma_j}(z)^2dz + \psi_{\gamma_j}(d)^2} \quad r_\gamma = \frac{1}{\pi}\frac{\sqrt{n_1^2k^2-\gamma}}{(n_1^2k^2-\gamma)\psi_\gamma(d)^2 + \partial_z\psi_\gamma(d)^2}.$$

- When $kD \gg 1$,

$$W_D^{\rm s}(x,\kappa;t,\omega) \simeq \sum_{j=0}^{N(\omega)-1} \frac{\partial_z \phi_j(0;\omega)^2}{\beta_j(\omega)} W_j(x;t,\omega)\delta(\kappa-\beta_j(\omega)) + \frac{\partial_z \tilde{\phi}_\xi(0;\omega)^2}{k\xi} \tilde{\mathcal{N}}(\xi)\tilde{W}_\xi(x;t,\omega)\big|_{\xi=\kappa/k},$$

and the Wigner transforms $(W_j, \tilde{W}_\xi)$ satisfy the coupled RTEs:

$$\partial_x W_j + \frac{1}{v_j}\partial_t W_j = \sum_{l=0,l\neq j}^{N-1} \Gamma_{jl} W_l + \int_0^\infty \tilde{\Gamma}_{j\xi'} \tilde{W}_{\xi'} \tilde{\mathcal{N}}(\xi')d\xi' - \Lambda_j W_j,$$

$$\partial_x \tilde{W}_\xi + \frac{1}{v_\xi}\partial_t \tilde{W}_\xi = \frac{1}{kD}\sum_{l=0}^{N-1} \tilde{\Gamma}_{\xi l} W_l + \frac{1}{kD}\int_0^\infty \tilde{\Gamma}_{\xi\xi'} \tilde{W}_{\xi'} \tilde{\mathcal{N}}(\xi')d\xi' - \frac{1}{kD}\tilde{\Lambda}_\xi \tilde{W}_\xi,$$

for $j = 0, \ldots, N-1$ and $\xi \in (0, n_1)$, to first order in $kD$, where

$$v_j(\omega) = \frac{1}{\partial_\omega \beta_j(\omega)}, \qquad v_\xi = \frac{c_o \xi}{n_1^2},$$

and the normalized density of states is $\tilde{\mathcal{N}}(\xi) = \frac{\xi}{\pi\sqrt{n_1^2-\xi^2}}\mathbf{1}_{(0,n_1)}(\xi)$,

the scattering coefficients are

$$\Gamma_{jl}(\omega) = \frac{k^4}{2\beta_j\beta_l(\omega)} \int_0^\infty \mathcal{R}_{jl}(x;\omega) \cos\left((\beta_l(\omega) - \beta_j(\omega))x\right) dx,$$

$$\tilde{\Gamma}_{j\xi}(\omega) = \frac{k^3}{2\beta_j(\omega)\xi} \int_0^\infty \tilde{\mathcal{R}}_{j\xi}(x;\omega) \cos\left((k\xi - \beta_j(\omega))x\right) dx,$$

$$\tilde{\Gamma}_{\xi\xi'}(\omega) = \frac{k^2}{2\xi\xi'} \int_0^\infty \tilde{\mathcal{R}}_{\xi\xi'}(x;\omega) \cos\left(k(\xi - \xi')x\right) dx,$$

the extinction coefficients are

$$\Lambda_j(\omega) = \sum_{l=0, l\neq j}^{N(\omega)-1} \Gamma_{jl}(\omega) + \int_0^\infty \tilde{\Gamma}_{j\xi'}(\omega)\tilde{\mathcal{N}}(\xi')d\xi',$$

$$\tilde{\Lambda}_\xi(\omega) = \sum_{l=0}^{N(\omega)-1} \tilde{\Gamma}_{\xi l}(\omega) + \int_0^\infty \tilde{\Gamma}_{\xi\xi'}(\omega)\tilde{\mathcal{N}}(\xi')d\xi',$$

the correlation functions are defined by

$$\mathcal{R}_{jl}(x;\omega) = \int_0^\infty \int_0^\infty \phi_j \phi_l(z;\omega) \mathbb{E}[\nu(0,z)\nu(x,z')] \phi_j \phi_l(z';\omega) dz dz',$$
$$\tilde{\mathcal{R}}_{j\xi}(x;\omega) = \int_0^\infty \int_0^\infty \phi_j \tilde{\phi}_\xi(z;\omega) \mathbb{E}[\nu(0,z)\nu(x,z')] \phi_j \tilde{\phi}_\xi(z';\omega) dz dz',$$
$$\tilde{\mathcal{R}}_{\xi\xi'}(x;\omega) = \int_0^\infty \int_0^\infty \tilde{\phi}_\xi \tilde{\phi}_{\xi'}(z;\omega) \mathbb{E}[\nu(0,z)\nu(x,z')] \tilde{\phi}_\xi \tilde{\phi}_{\xi'}(z';\omega) dz dz',$$

where for $\xi \in (0, n_1)$,

$$\tilde{\phi}_\xi(z;\omega) = \frac{\sqrt{2\xi}}{\sqrt{\tilde{\mathcal{N}}(\xi)}} \phi_{k^2\xi^2}(z;\omega),$$

with $\phi_j$, $j = 0, \ldots, N-1$, a mode corresponding to the discrete spectrum of the half-space problem,

$\phi_\gamma$, $\gamma = k^2\xi^2$, a mode corresponding to the continuous spectrum of the half-space problem.

- The mean mode powers,

$$P_j(x;\omega) = \int_{-\infty}^{\infty} W_j(x;t,\omega)dt, \qquad \tilde{P}_\xi(x;\omega) = \int_{-\infty}^{\infty} \tilde{W}_\xi(x;t,\omega)dt,$$

satisfy

$$\partial_x P_j = \sum_{l=0, l\neq j}^{N-1} \Gamma_{jl} P_l + \int_0^\infty \tilde{\Gamma}_{j\xi'} \tilde{P}_{\xi'} \tilde{\mathcal{N}}(\xi')d\xi' - \Lambda_j P_j,$$

$$\partial_x \tilde{P}_\xi = \frac{1}{kD} \sum_{l=0}^{N-1} \tilde{\Gamma}_{\xi l} P_l + \frac{1}{kD} \int_0^\infty \tilde{\Gamma}_{\xi\xi'} \tilde{P}_{\xi'} \tilde{\mathcal{N}}(\xi')d\xi' - \frac{1}{kD} \tilde{\Lambda}_\xi \tilde{P}_\xi.$$

The total power,

$$\mathcal{P}(x;\omega) = \sum_{j=0}^{N(\omega)-1} P_j(x;\omega) + kD \int_0^\infty \tilde{P}_\xi(x;\omega) \tilde{\mathcal{N}}(\xi)d\xi,$$

is a conserved quantity, that is, $\partial_x \mathcal{P} = 0$.

• The parameters $\Gamma$ and $\tilde{\Gamma}$ are of order $k^2\sigma^2\ell_c$ where $\sigma$ and $\ell_c$ are the standard deviation and the correlation length (in $x$) of the random fluctuations $\nu(x,z)$.

Assume a source that generates surface waves.

For propagation distances of the order of $1/(k^2\sigma^2\ell_c)$, the RTE for the surface modes can be reduced to

$$\partial_x W_j + \frac{1}{v_j}\partial_t W_j = \sum_{l=0, l\neq j}^{N-1} \Gamma_{jl} W_l - \Lambda_j W_j,$$

which is the RTE determined in the half-space case.

• The power is initially carried by the surface modes. Coupling induces changes in the distribution amongst the surface modes and a decay, which gives for $x$ of the order of $1/(k^2\sigma^2\ell_c)$,

$$\big(P_j(x)\big)_{j=0}^{N-1} \simeq \exp(-\mathbf{M}x)\big(P_j(0)\big)_{j=0}^{N-1},$$

where $\mathbf{M}$ is the positive matrix with entries

$$M_{jl} = \Lambda_j \delta_{jl} - \Gamma_{jl}.$$

The decay is in fact a transfer of power from the surface modes to the body modes, expressed by

$$\partial_x \tilde{P}_\xi = \frac{1}{kD} \sum_{l=0}^{N-1} \tilde{\Gamma}_{\xi l} P_l, \quad \tilde{P}_\xi(0) = 0,$$

which gives for $x$ of the order of $1/(k^2\sigma^2\ell_c)$,

$$\tilde{P}_\xi(x) = \frac{1}{kD} \sum_{l,l'=0}^{N-1} \tilde{\Gamma}_{\xi l} \big(\mathbf{M}^{-1}(\mathbf{I} - \exp(-\mathbf{M}x)\big)_{ll'} P_{l'}(0).$$

• For propagation distances $x \sim D/(k\sigma^2\ell_c)$ the mean surface and body mode powers become of the order of $\mathcal{P}(0)/(kD)$ with $\mathcal{P}(0) = \sum_{j=0}^{N-1} P_j(0)$.

The mean surface mode powers are in a quasi-equilibrium state that is determined by the mean body mode power distribution:

$$(P_j(x))_{j=0}^{N-1} = \mathbf{M}^{-1} \left( \int_0^\infty \tilde{\Gamma}_{j\xi'} \tilde{P}_{\xi'}(x) \tilde{\mathcal{N}}(\xi') d\xi' \right)_{j=0}^{N-1}.$$

The mean body mode powers $\tilde{P}_\xi$ slowly evolve at the scale $D/(k\sigma^2\ell_c)$:

$$\begin{aligned} \partial_x \tilde{P}_\xi =& \frac{1}{kD} \sum_{l,l'=0}^{N-1} \int_0^\infty \tilde{\Gamma}_{\xi l} (\mathbf{M}^{-1})_{ll'} \tilde{\Gamma}_{l'\xi'} \tilde{P}_{\xi'} \tilde{\mathcal{N}}(\xi') d\xi' \\ &+ \frac{1}{kD} \int_0^\infty \tilde{\Gamma}_{\xi\xi'} \tilde{P}_{\xi'} \tilde{\mathcal{N}}(\xi') d\xi' - \frac{1}{kD} \tilde{\Lambda}_\xi \tilde{P}_\xi, \end{aligned}$$

starting from $\tilde{P}_{\mathrm{ini},\xi} := \dfrac{1}{kD} \displaystyle\sum_{l,l'=0}^{N-1} \tilde{\Gamma}_{\xi l} (\mathbf{M}^{-1})_{ll'} P_{l'}(0)$.

First term: conversion from a body mode $\xi'$ to a body mode $\xi$ mediated by surface modes $\xi' \mapsto l' \mapsto l \mapsto \xi$

Second term: direct conversion from a body mode $\xi'$ to a body mode $\xi$.

• The equipartition principle takes, here, the following form: As $x \to +\infty$ ($x \gg D/(k\sigma^2\ell_c)$), $P_j(x)$ and $\tilde{P}_\xi(x)$ converge to

$$\mathcal{P}_\infty(\omega) = \frac{\mathcal{P}(0;\omega)}{kD\int_0^\infty \tilde{\mathcal{N}}(\xi)d\xi} = \frac{\pi\mathcal{P}(0;\omega)}{n_1 kD}.$$

$\hookrightarrow$ Most of the power is carried by body modes (the fraction of power carried by the surface modes is of the order of $d/D$).

• The spatial profiles $\phi_{D,\gamma}$ of the body waves extend throughout $(0, D)$ while the profiles $\phi_{j,D}$ of the surface waves are concentrated on $(0, d)$.
$\hookrightarrow$ The contributions of the body waves and of the surface waves to $W^{\rm s}$ are of the same order:

$$\int_{-\infty}^{+\infty} W_D^{\rm s}(x,\kappa;t,\omega)dt \simeq \mathcal{P}_\infty(\omega)\sum_{j=0}^{N(\omega)-1}\frac{\partial_z\phi_j(0;\omega)^2}{\beta_j(\omega)}\delta(\kappa-\beta_j(\omega)) + \mathcal{P}_\infty(\omega)\frac{\partial_z\tilde{\phi}_\xi(0;\omega)^2}{k\xi}\tilde{\mathcal{N}}(\xi)\big|_{\xi=\kappa/k},$$

for propagation distances $x$ larger than $D/(k\sigma^2\ell_c)$.

# Conclusions and perspectives

- Two-dimensional scalar toy model for seismic coda: a randomly heterogeneous half-space or thick waveguide that has a thin layer beneath the surface that supports a finite number of guided (surface) modes.
- An original RTE that involves:
  - a slowly evolving metastable surface mode distribution (which is not the equipartitioned distribution) and ultimately leads to energy equipartition between all modes.
  - a nontrivial coupling mechanism between body modes mediated by surface modes.
- Next steps:
  - analyze the associated inverse problem, addressing the claim that the background index of refraction can be robustly determined from the Wigner transform or related albedo operator.
  - address the three-dimensional, elastic system.