On the stability of black hole quasi-normal modes: a pseudospectrum approach

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(based on joint work with Rodrigo P. Macedo, Lamis Al Sheikh and Edgar Gasperín)

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Resonances, Inverse Problems and Seismic Waves 2021
LMR, Reims, 17 November 2021
1. The Problem in a nutshell: (asymptotically flat) BH QNM instability

2. Non-normal operators: spectral instability and Pseudospectrum

3. Hyperboloidal approach to QNMs: the Pöschl-Teller toy model

4. Black Hole QNM ultraviolet instability

5. Discussion, Conclusions and Perspectives: a low-regularity problem
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The Problem in a nutshell: (asymptotically flat) BH QNM instability

Black Hole QNM instabilities [cf. Nollert 96, Nollert & Price 99]

**Perturbation theory on a Schwarzschild Black Hole: spherically symmetric case**

Scalar, electromagnetic and gravitational perturbations reduced to (Minkowski) 1+1 wave equation for \( \phi_{\ell m}(t, r_*) \) with a potential \( V_\ell \) [Regge-Wheeler 57, Zerilli 70]:

\[
\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V_\ell \right) \phi_{\ell m} = 0 \quad , \quad t \in ] - \infty, \infty[ , \quad r_* \in ] - \infty, \infty[
\]

**Schwarzschild quasi-normal modes**

Convention for a “mode”: \( \phi_{\ell m}(t, r_*) \sim e^{i \omega t} \hat{\phi}_{\ell m}(r_*) \).

“Spectral” problem with “outgoing boundary” conditions:

\[
\left( - \frac{\partial^2}{\partial r_*^2} + V_\ell \right) \hat{\phi}_{\ell m} = \omega^2 \hat{\phi}_{\ell m} \quad , \quad r_* \in ] - \infty, \infty[ \\
\hat{\phi}_{\ell m} \sim e^{-i \omega r_*} \quad , \quad (r_* \to \infty) \quad , \quad \hat{\phi}_{\ell m} \sim e^{i \omega r_*} \quad , \quad (r_* \to -\infty)
\]

Time evolution stability: \( \text{Im}(\omega) > 0 \). Exponential divergence of \( \hat{\phi}_{\ell m} \) at \( \pm \infty \).
Schwarzschild gravitational QNMs

Schwarzschild QNMs ($\ell = 2$ diamonds, $\ell = 3$ crosses) [e.g. Kokkotas & Schmidt 99] 
QNM frequencies $\omega_n$ are invariant probes into the background geometry
The Problem in a nutshell: (asymptotically flat) BH QNM instability

Black Hole QNM instabilities [cf. Nollert 96, Nollert & Price 99]

Nollert’s work on stair-case discretizations of Schwarzschild
(revisited in [Daghigh, Green & Morey 20, arXiv:2002.0725])

- Instability of the slowest decaying QNM (but ringdown “stability”).
- Instabilities of ”highly damped QNMs”.

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On the stability of black hole quasi-normal modes
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Posing the problem

Consider the operator on functions (defined on “appropriate” functional spaces) with non-compact-domain and with $V > 0$ with appropriate decay at infinity:

$$P_V = -\Delta + V$$

QNMs in the theory of Scattering Resonances

[Lax & Phillips, Vainberg; Sjöstrand, Zworski, Petkov, Iantchenko, ...many others; e.g. Dyatlov & Zworski 20]

Resolvent $R_V(\lambda) = (P_V - \lambda)^{-1}$ analytic $\text{Im}(\lambda) > 0$. Scattering resonances: poles of the meromorphic extension of (truncated) $R_V(\lambda)$ to $\text{Im}(\lambda) < 0$.

QNMs as a “proper” eigenvalue problem: **non-selfadjoint operators**

- “Complex scaling” [Simon 78, Reed & Simon 78, Sjöstrand...]: not the approach followed here.
- Hyperboloidal approach [Friedman & Schutz 75, Schmidt 93, Bizon, Zenginoglu 11, Vasy 13, Warnick 15, Ansorg & P.-Macedo 16, Gajic & Warnick 19, Bizon et al. 20, Galkowski & Zworski 20, ...]

Problem in terms of “eigenvalue problem” of **non-selfadjoint operator** $L$:

$$L u_{\ell m} = \omega u_{\ell m}, \quad u_{\ell m} \in H \text{ (Hilbert space)}$$

- **Geometric boundary conditions**: Null infinity reached by hyperboloidal slices.
- **Regularity conditions on** $u_{\ell m}$: choice of appropriate $H$, then $\omega \in \sigma_p(L)$. 
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5. Discussion, Conclusions and Perspectives: a low-regularity problem
Normal operators: Spectral Theorem

- **Normality**: denoting the adjoint matrix by $L^\dagger$, then $L$ is normal iff
  \[
  [L, L^\dagger] = LL^\dagger - L^\dagger L = 0
  \]

  Matrix examples: symmetric, hermitian, orthogonal, unitary...

- **Spectral Theorem** (”moral statement”):
  $L$ is normal iff is unitarily diagonalisable.

Note: this depends on the adjoint $L^\dagger$, then on the Hilbert space (scalar product).
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'Non-normal' operators, $[L, L^\dagger] \neq 0$: no Spectral Theorem

- Completeness more difficult to study.
- Eigenvectors not necessarily orthogonal.
- **Spectral instabilities**.
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Methodology here: "exploration" stage

**Numerical spectral methods**: Chebyshev polynomials truncations.
**Spectral Theorem. Normal and ’non-normal’ operators**

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**Example of spectral instability**

\[ L = a \frac{d^2}{dx^2} + b \frac{d}{dx} + c \quad , \quad a, b, c \in \mathbb{R} \]

acting on functions in $L^2([0,1])$, with homogeneous Dirichlet conditions (Chebyshev finite-dimensional matrix approximates).
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Example of spectral instability

\[ L = a \frac{d^2}{dx^2} + b \frac{d}{dx} + c + \epsilon E_{\text{Random}} \quad , \quad a, b, c \in \mathbb{R}, \quad \|E_{\text{Random}}\| = 1 \]

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![](image)

Eigenvalues of $L$ with $n = N + 1 = 51$ points

\[ a = -1, \ b = 0, \ c = 1, \ \epsilon = 10^{-2} \]
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\[
\begin{align*}
a &= -1, & b &= 30, & c &= 1, & \epsilon &= 0
\end{align*}
\]
Non-normal operators: spectral instability and Pseudospectrum

Spectral Theorem. Normal and 'non-normal' operators

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Eigenvalues of $L$ with $n = N + 1 = 51$ points

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Eigenvalues of $L$ with $n = N + 1 = 51$ points

\[ a = -1, \quad b = 30, \quad c = 1, \quad \epsilon = 10^{-8} \]
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Eigenvalues of $L$ with $n = N + 1 = 51$ points

$a = -1, \ b = 30, \ c = 1, \ \epsilon = 10^{-4}$

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Non-normal operators: spectral instability and Pseudospectrum

**Spectral Theorem. Normal and 'non-normal' operators**

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\[a = -1, \quad b = 30, \quad c = 1, \quad \epsilon = 10^{-2}\]
Non-normal operators: spectral instability and Pseudospectrum

Spectral (in)stability: eigenvalue condition number

Left- and right-eigenvectors, respectively $u_i$ and $v_i$, of $A$

$$A^\dagger u_i = \bar{\lambda}_i u_i \quad (\Leftrightarrow u_i^\dagger A = \lambda_i u_i^\dagger) \quad , \quad Av_i = \lambda_i v_i \quad , \quad i \in \{1, \ldots, n\} ,$$

Perturbation theory of eigenvalues [cf. Kato 80, ...; e.g. Trefethen, Embree 05]:

$$A(\epsilon) = A + \epsilon \delta A \quad , \quad ||\delta A|| = 1 .$$

$$|\lambda_i(\epsilon) - \lambda_i| = \epsilon \frac{|\langle u_i, \delta A v_i(\epsilon) \rangle|}{|\langle u_i, v_i \rangle|} \leq \epsilon \frac{||u_i|| ||\delta A v_i||}{|\langle u_i, v_i \rangle|} + O(\epsilon^2) \leq \epsilon \frac{||u_i|| ||v_i||}{|\langle u_i, v_i \rangle|} + O(\epsilon^2).$$

Eigenvalue condition number: $\kappa(\lambda_i)$

$$\kappa(\lambda_i) = \frac{||u_i|| ||v_i||}{|\langle u_i, v_i \rangle|}$$
Given $\epsilon > 0$, the $\epsilon$-pseudospectrum $\sigma_\epsilon(L)$ of $L$ is defined as [e.g Trefethen & Embree 05]:

$$\sigma_\epsilon(L) = \{ \lambda \in \mathbb{C}, \text{ such that } \lambda \in \sigma(L + \delta L) \text{ for some } \delta L \text{ with } ||\delta L|| < \epsilon \}$$

$$= \{ \lambda \in \mathbb{C}, \text{ such that } ||Lv - \lambda v|| < \epsilon \text{ for some } v \text{ with } ||v|| = 1 \}$$

$$= \{ \lambda \in \mathbb{C}, \text{ such that } ||(\lambda I - L)^{-1}|| > \epsilon^{-1} \}$$
Spectral (in)stability and Pseudospectrum

Pseudospectrum

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$$

Normal case: bounds on the norm of the resolvent $R_L(\lambda) = (\lambda I - L)^{-1}$

Given $\lambda \in \mathbb{C}$ and $\sigma(L)$ the spectrum of $L$, it holds

$$
||((\lambda I - L)^{-1})||_2 = \frac{1}{\text{dist}(\lambda, \sigma(L))}
$$
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$$

Non-normal case: bad control on the resolvent $R_L(\lambda)$. **Pseudospectrum**

The norm of the resolvent can become very large far from the spectrum:

$$
|| (\lambda I - L)^{-1} ||_2 \leq \frac{\kappa}{\text{dist}(\lambda, \sigma(L))}
$$

where $\kappa$ is a “condition number” assessing the lack of proportionality of 'left' and 'right' eigenvectors of $L$, and can become very large in the non-normal case.
Pseudospectrum and condition number: Bauer-Fike theorem [e.g Trefethen & Embree 05]

Defining “tubular neighbourhood” of radius $\epsilon$ around $\sigma(A)$

$$\Delta_\epsilon(A) = \{ \lambda \in \mathbb{C} : \text{dist} (\lambda, \sigma(A)) < \epsilon \} ,$$

it holds: $\Delta_\epsilon(A) \subseteq \sigma_\epsilon(A)$. For normal operators: $\sigma_\epsilon(A) = \Delta_\epsilon(A)$.

Non-normal case, $\kappa(\lambda_i) \neq 1$, it holds (for small $\epsilon$):

$$\sigma_\epsilon(A) \subseteq \bigcup_{\lambda_i \in \sigma(A)} \Delta_{\epsilon \kappa(\lambda_i) + O(\epsilon^2)}(\{\lambda_i\}) ,$$

Therefore $\sigma_\epsilon(A)$ larger tubular neighbourhood of radius $\sim \epsilon \kappa(\lambda_i)$. 
Pseudospectrum and condition number: Bauer-Fike theorem

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Random perturbations and Pseudospectrum

Random perturbations $\Delta L$ with $||\delta L|| < \epsilon$ “push” eigenvalues into $\sigma_\epsilon(A)$, providing an insightful and systematic manner of exploring $\sigma_\epsilon(L)$. 
Non-normal operators: spectral instability and Pseudospectrum

Bauer-Fike theorem. Random perturbations

Pseudospectrum and condition number: Bauer-Fike theorem [e.g Trefethen & Embree 05]

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The ’role’ of random perturbations [Sjöstrand 19; Hager 05, Montrieux, Nonnenmacher, Vogel,...]

Random perturbations improve the analytical behaviour of $R_L(\lambda)$!!!
Pseudospectrum of: $L = a \frac{d^2}{dx^2} + b \frac{d}{dx} + c + \epsilon E_{\text{Random}}$

Spectrum and Pseudospectrum of $L$ with $\log ||\text{Random}||_2 = -50$

- $a = -1$, $b = 0$, $c = 1$, $\epsilon = 0$
Pseudospectrum of: \( L = a \frac{d^2}{dx^2} + b \frac{d}{dx} + c + \epsilon E_{\text{Random}} \)

Spectrum and Pseudospectrum of \( L \) with \( \log ||\text{Random}||_2 = 1 \)

\( a = -1, \ b = 0, \ c = 1, \ \epsilon = 10^1 \)
Pseudospectrum of: \( L = a \frac{d^2}{dx^2} + b \frac{d}{dx} + c + \epsilon E_{\text{Random}} \)

Spectrum and Pseudospectrum of \( L \) with \( \log||\text{Random}||_2 = -15 \)

\[
\begin{align*}
\text{Re}(\omega_n) & \quad 0 \quad 10^2 \quad 2 \times 10^2 \quad 3 \times 10^2 \quad 4 \times 10^2 \quad 5 \times 10^2 \\
\text{Im}(\omega_n) & \quad -2 \times 10^2 \quad -1.5 \times 10^2 \quad -1 \times 10^2 \quad -1.5 \times 10^2 \quad -2 \times 10^2
\end{align*}
\]
Pseudospectrum of: \( L = a \frac{d^2}{dx^2} + b \frac{d}{dx} + c + \epsilon E_{\text{Random}} \)

Spectrum and Pseudospectrum of \( L \) with \( \log ||\text{Random}||_2 = -10 \)
Pseudospectrum of: \[ L = a \frac{d^2}{dx^2} + b \frac{d}{dx} + c + \epsilon E_{\text{Random}} \]

Spectrum and Pseudospectrum of \( L \) with \( \log ||\text{Random}||_2 = -8 \)

\[ a = -1, \ b = 30, \ c = 1, \ \epsilon = 10^{-8} \]
Non-normal operators: spectral instability and Pseudospectrum

Spectral (in)stability and Pseudospectrum: illustration

Pseudospectrum of: $L = a \frac{d^2}{dx^2} + b \frac{d}{dx} + c + \epsilon E_{\text{Random}}$

Spectrum and Pseudospectrum of $L$ with $\log ||\text{Random}||_2 = -6$

$a = -1$, $b = 30$, $c = 1$, $\epsilon = 10^{-6}$
Pseudospectrum of: \( L = a \frac{d^2}{dx^2} + b \frac{d}{dx} + c + \epsilon E_{\text{Random}} \)

Spectrum and Pseudospectrum of \( L \) with \( \log ||\text{Random}||_2 = -4 \)

\( a = -1, \ b = 30, \ c = 1, \ \epsilon = 10^{-4} \)
Pseudospectrum of: \[ L = a \frac{d^2}{dx^2} + b \frac{d}{dx} + c + \epsilon E_{\text{Random}} \]

Spectrum and Pseudospectrum of \( L \) with \( \log ||\text{Random}||_2 = -2 \)

\[ a = -1, \ b = 30, \ c = 1, \ \epsilon = 10^{-2} \]
The relevance of the scalar product: assessing large/small

The illustrative operator: \( L = a \frac{d^2}{dx^2} + b \frac{d}{dx} + c \), \( a, b, c \in \mathbb{R} \)

- Non-selfadjoint in standard \( L^2([0, 1]) \) for \( b \neq 0 \).
- Formally normal!
- Non-normal: domain of \( L^\dagger L \) and \( LL^\dagger \) different.
- But actually self-adjoint...

Cast in Sturm-Liouville form: selfadjoint for appropriate scalar product \( \langle \cdot, \cdot \rangle_w \) !!!

Pseudospectrum using the \( L^2 \)-inner-product
Non-normal operators: spectral instability and Pseudospectrum

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Cast in Sturm-Liouville form: selfadjoint in appropriate scalar product \( \langle \cdot, \cdot \rangle_w \)

Pseudospectrum using Gram Matrix = SturmLiouville-w

![Graph showing pseudospectrum](image)
Scheme

1. The Problem in a nutshell: (asymptotically flat) BH QNM instability
2. Non-normal operators: spectral instability and Pseudospectrum
3. Hyperboloidal approach to QNMs: the Pöschl-Teller toy model
4. Black Hole QNM ultraviolet instability
5. Discussion, Conclusions and Perspectives: a low-regularity problem
Hyperboloidal approach to QNMs

- **Spectral problem**: homogeneous wave equation with purely outgoing boundary conditions.
- Outgoing BCs naturally imposed at \( \mathcal{I}^+ \).
- Outgoing BCs actually “incorporated” at \( \mathcal{I}^+ \):
  - Geometrically: null cones outgoing.
  - Analytically: BCs encoded into a singular operator, ”**BCs as regularity conditions**”.
- **Eigenfunctions** do not diverge when \( x \to \infty \): actually **integrable**. Key to Hilbert space.
Hyperboloidal approach to QNMs

- B. Schmidt [Schmidt 93; cf. also Friedman & Schutz 75]
- Analysis in the conformally compactified picture [Friedrich; Frauendiener,...]
- Framework for BH perturbations [Zenginoglu 11].
- QNM definition as operator eigenvalues [Bizoń...; Bizoń, Chmaj & Mach 20].
- QNMs of asymp. AdS spacetimes [Warnick 15].
- Schwarzschild QNMs [Ansorg & Macedo 16]. (cf. also Reissner-Nordström [Macedo, JLJ, Ansorg 18]).
- ”Gevrey” [Gajic & Warnick 19; Galkowski & Zworski 20].
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- "Gevrey" [Gajic & Warnick 19; Galkowski & Zworski 20].

1. Wave problem in spherically symmetric asymptotically flat case

As starting point, consider the problem for a $\phi_{\ell m}$ mode in tortoise coordinates:

$$\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^*_2} + V_\ell \right) \phi_{\ell m} = 0 \quad , \quad t \in ]-\infty, \infty[, \quad r^* \in ]-\infty, \infty[$$
2. Choice of hyperboloidal foliation and compactification

Make the change to Bizoń-Mach variables [Bizoń & Mach 17]:

\[
\begin{align*}
\tau &= t - \ln (\cosh r_*) \\
x &= \tanh r_*
\end{align*}
\]

, \( \tau \in ]-\infty, \infty[ \), \( x \in ]-1, 1[ \)

1. \( \tau = \text{const.} \) defines a hyperboloidal slicing.
2. Compactification along hyperboloidal slices.
2. Choice of hyperboloidal foliation and compactification

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1. \( \tau = \text{const.} \) defines a hyperboloidal slicing.
2. Compactification along hyperboloidal slices.
3. We add the boundaries \( y = \pm 1 \).
Compactification along hyperboloidal slices

2. Choice of hyperboloidal foliation and compactification

Make the change to Bizoń-Mach variables \([\text{Bizoń & Mach 17}]:\)

\[
\begin{align*}
\tau &= t - \ln (\cosh r_*) \\
x &= \tanh r_*
\end{align*}
\]

1. \(\tau = \text{const.}\) defines a hyperboloidal slicing.
2. Compactification along hyperboloidal slices.
3. We add the boundaries \(y = \pm 1\).

3. Wave equation in hyperboloidal coordinates: no boundary conditions allowed

For \(x = \pm 1, V_\ell = 0\). In the interior, \(x \in ] - 1, 1[\):

\[
\left( \frac{\partial^2}{\partial \tau^2} + 2x \frac{\partial}{\partial \tau} \frac{\partial}{\partial x} + \frac{\partial}{\partial \tau} + 2x \frac{\partial}{\partial x} - (1 - x^2) \frac{\partial^2}{\partial x^2} + \tilde{V}_\ell \right) \phi_{\ell m} = 0 ,
\]

with \(\tilde{V}_\ell = \frac{V_\ell}{(1 - x^2)}\).
4. Evolution equation in first order form

Introducing the auxiliary field

\[ \psi_{\ell m} = \partial_\tau \phi_{\ell m}, \]

we can write the wave equation in first-order form:

\[
\begin{pmatrix}
\partial_\tau \\
\phi_{\ell m} \\
\psi_{\ell m}
\end{pmatrix}
= \begin{pmatrix}
0 & 1 \\
(1 - x^2) \partial_x^2 - 2x \partial_x - \tilde{V}_{\ell m} & -(2x \partial_y + 1)
\end{pmatrix}
\begin{pmatrix}
\phi_{\ell m} \\
\psi_{\ell m}
\end{pmatrix}.
\]
5. Eigenvalue problem for a non-selfadjoint operator, no BCs

Our problem: study

\[ L \begin{pmatrix} \phi_{\ell m} \\ \psi_{\ell m} \end{pmatrix} = \omega \begin{pmatrix} \phi_{\ell m} \\ \psi_{\ell m} \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 1 \\ L_1 & L_2 \end{pmatrix}, \]

where

\[ L_1 = (1 - x^2)\partial_x^2 - 2x\partial_x - \tilde{V}_{\ell m}, \quad L_2 = -(2x\partial_x + 1). \]
Spectral problem: first order formulation

5. Eigenvalue problem for a non-selfadjoint operator, no BCs

Our problem: study

$$\hat{L} \begin{pmatrix} \phi_{\ell m} \\ \psi_{\ell m} \end{pmatrix} = \omega \begin{pmatrix} \phi_{\ell m} \\ \psi_{\ell m} \end{pmatrix} , \quad L = \begin{pmatrix} 0 & 1 \\ L_1 & L_2 \end{pmatrix} ,$$

where

$$L_1 = \partial_x \left( (1 - x^2) \partial_x \right) - \tilde{V}_{\ell m} , \quad L_2 = -(2\Omega \cdot \nabla + \text{div}\Omega) \quad (\text{with } \Omega = x).$$
Spectral problem: first order formulation

5. Eigenvalue problem for a non-selfadjoint operator, no BCs

Our problem: study

\[ \hat{L} \left( \begin{array}{c} \phi_{\ell m} \\ \psi_{\ell m} \end{array} \right) = \omega \left( \begin{array}{c} \phi_{\ell m} \\ \psi_{\ell m} \end{array} \right), \quad L = \left( \begin{array}{c} 0 \\ L_1 \\ L_2 \end{array} \right), \]

where

\[ L_1 = \partial_x \left( (1 - x^2) \partial_x \right) - \tilde{V}_{\ell m}, \quad L_2 = -(2\Omega \cdot \nabla + \text{div}\Omega) \quad \text{(with} \quad \Omega = x). \]

QNM problem as a “proper” eigenvalue problem. But...Hilbert space?

Spectral problem in a Hilbert space with “Energy” scalar product \((\tilde{V} > 0)\):

\[ \langle \left( \begin{array}{c} \phi_1 \\ \psi_1 \end{array} \right), \left( \begin{array}{c} \phi_2 \\ \psi_2 \end{array} \right) \rangle_E = \int_{\Sigma_T} \left( \bar{\psi}_1 \psi_2 + (1 - x^2) \partial_x \bar{\phi}_1 \partial_x \phi_2 + \tilde{V} \bar{\phi}_1 \phi_2 \right) d\Sigma_t \]
1. The Problem in a nutshell: (asymptotically flat) BH QNM instability

2. Non-normal operators: spectral instability and Pseudospectrum

3. Hyperboloidal approach to QNMs: the Pöschl-Teller toy model

4. Black Hole QNM ultraviolet instability

5. Discussion, Conclusions and Perspectives: a low-regularity problem
Black Hole QNM instabilities \[\text{[cf. Nollert 96, Nollert & Price 99]}\]

Schwarzschild gravitational QNMs

Schwarzschild QNMs \((\ell = 2 \text{ diamonds}, \ell = 3 \text{ crosses})\) \[\text{[e.g. Kokkotas & Schmidt 99]}\]
Instability of the slowest decaying QNM (but ringdown “stability”).

Instabilities of "highly damped QNMs".

Various interests in BH QNM perturbations:

i) “Dirty” astrophysical black holes [Leung et al. 97; Barausse, Cardoso & Pani 14;...]

ii) Quantum (highly damped QNMs/high frequency instability) [Hod 98; Maggiore 08; Babb, Daghigh & Kunstatter 11; Ciric, Konjik & Samsarov 19, Olmedo; ...].
Test-bed study: Pöschl-Teller potential

Pöschl-Teller potential [JLJ, Macedo & Al Sheikh 21] (toy-model in [Bizoń, Chmaj & Mach 20])

\[ V = V_o \, \text{sech}^2(r_*) = V_o (1 - y^2) \implies \tilde{V} = V_o \]

Particularly simple form (scalar field in de Sitter, \( m^2 = V_o \) [Bizoń, Chmaj & Mach 20])

- Integrable potential (QNM completeness [Beyer 99] with \( m^2 = V_o \)).
- QNM frequencies: \( \omega_n^\pm = -is_n^\pm = \pm \frac{\sqrt{3}}{2} + i \left( n + \frac{1}{2} \right) \)
- Here, eigenfunctions are Jacobi polynomials: \( \phi_n(y) = P_n^{(s_n^\pm,s_n^\pm)}(y) \).

Pöschl-Teller QNM Perturbed-Spectra: Random Potential

\[ ||\delta V_r|| = 0 \]
Test-bed study: Pöschl-Teller potential

Pöschl-Teller potential \[\text{[JLJ, Macedo & Al Sheikh 21]}\text{ (toy-model in [Bizoń, Chmaj & Mach 20])}\]

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\[\|\delta V_r\| = 10^{-16} \times\]
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  \[
  \omega_n^\pm = -is_n^\pm = \pm \frac{\sqrt{3}}{2} + i \left( n + \frac{1}{2} \right)
  \]
- Here, eigenfunctions are Jacobi polynomials:
  \[\phi_n(y) = P_n(s_n^\pm, s_n^\pm)(y)\].

\[||\delta V_r|| = 10^{-8}\]
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- Here, eigenfunctions are Jacobi polynomials: \( \phi_n(y) = P_n^{(s^+_n, s^-_n)}(y) \).

![Pöschl-Teller QNM Perturbed-Spectra: Deterministic Potential](image-url)
Test-bed study: Pöschl-Teller potential

Pöschl-Teller potential [JLJ, Macedo & Al Sheikh 21] (toy-model in [Bizoń, Chmaj & Mach 20])

\[ V = V_o \text{sech}^2(r_*) = V_o (1 - y^2) \implies \tilde{V} = V_o \]

Particularly simple form (scalar field in de Sitter, \(m^2 = V_o\) [Bizoń, Chmaj & Mach 20])

- Integrable potential (QNM completeness [Beyer 99] with \(m^2 = V_o\) !).
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![Graph of QNM frequencies with \( ||\delta V_d|| = 10^{-8} \) and \( k = 10 \).](image)
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- QNM frequencies: \( \omega_n^\pm = -is_n^\pm = \pm \frac{\sqrt{3}}{2} + i \left( n + \frac{1}{2} \right) \)
- Here, eigenfunctions are Jacobi polynomials: \( \phi_n(y) = P_n(s_n^\pm, s_n^\pm)(y) \).

\[ ||\delta V_d|| = 10^{-8}, k = 20 \]

\[ \text{Re} (\omega_n), \text{Im} (\omega_n) \]
Test-bed study: Pöschl-Teller potential

Pöschl-Teller potential [JLJ, Macedo & Al Sheikh 21] (toy-model in [Bizoń, Chmaj & Mach 20])

\[ V = V_o \text{ sech}^2(r_*) = V_o (1 - y^2) \quad \Rightarrow \quad \tilde{V} = V_o \]

Particularly simple form (scalar field in de Sitter, \( m^2 = V_o \) [Bizoń, Chmaj & Mach 20])

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- Here, eigenfunctions are Jacobi polynomials: \( \phi_n(y) = P_n(s_n^\pm, s_n^\pm)(y) \).

Poschel-teller Spectrum and Pseudospectrum of \( L \)
Test-bed study: Pöschl-Teller potential

Consistency check: self-adjoint case $\hat{L}_2 = 0$.

Spectrum and Pseudospectrum of $L$ with $\log||\text{Random}||_2 = -50$

$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\epsilon = 0$$
Consistency check: self-adjoint case $\hat{L}_2 = 0$.

Spectrum and Pseudospectrum of $L$ with $\log \|\text{Random}\|_2 = -5$

$$\tilde{V} = \tilde{V}_{\text{Pöschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$\epsilon = 10^{-5}$
Test-bed study: Pöschl-Teller potential

Consistency check: self-adjoint case \( \hat{L}_2 = 0 \).

\[
\text{Spectrum and Pseudospectrum of } L \text{ with } \log ||\text{Random}||_2 = -1
\]

\[
\tilde{V} = \tilde{V}_{\text{Pöschl–Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1
\]

\[
\epsilon = 10^{-1}
\]
Consistency check: self-adjoint case $\hat{L}_2 = 0$.

Spectrum and Pseudospectrum of $L$ with $log||\text{Random}||_2 = 0$

$$\tilde{V} = \tilde{V}_{\text{Poeschl–Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\epsilon = 10^0$$
Test-bed study: Pöschl-Teller potential

QNM problem: \( \hat{L}_2 \neq 0 \).

Poschel-teller Spectrum and Pseudospectrum of \( L \)
Test-bed study: Pöschl-Teller potential


Spectrum and Pseudospectrum of $L$ with $\log ||\text{Random}||_2 = -15$

\[
\tilde{V} = \tilde{V}_{\text{Pöschl–Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1
\]

$\epsilon = 10^{-15}$
Test-bed study: Pöschl-Teller potential


Spectrum and Pseudospectrum of $L$ with $\log||\text{Random}||_2 = -14$

$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\epsilon = 10^{-14}$$
Test-bed study: Pöschl-Teller potential


Spectrum and Pseudospectrum of \( L \) with \( \log ||\text{Random}||_2 = -13 \)

\[
\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1
\]

\( \epsilon = 10^{-13} \)
Test-bed study: Pöschl-Teller potential


\[ \tilde{V} = \tilde{V}_{\text{Pöschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1 \]

\[ \epsilon = 10^{-12} \]
Test-bed study: Pöschl-Teller potential


Spectrum and Pseudospectrum of $L$ with $\log||Random||_2 = -11$

\[ \tilde{V} = \tilde{V}_{\text{Pöschl–Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1 \]

\[ \epsilon = 10^{-11} \]
Test-bed study: Pöschl-Teller potential


Spectrum and Pseudospectrum of $L$ with $\log||\text{Random}||_2 = -10$

\[ \tilde{V} = \tilde{V}_{\text{Pöschl–Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1 \]

\[ \epsilon = 10^{-10} \]
Test-bed study: Pöschl-Teller potential


\[ \tilde{V} = \tilde{V}_{\text{Pöschl–Teller}} + \epsilon \|E_{\text{Random}}\|, \quad \|E_{\text{Random}}\| = 1 \]

\[ \epsilon = 10^{-9} \]
Test-bed study: Pöschl-Teller potential


\[ \tilde{V} = \tilde{V}_{\text{Pöschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1 \]

\[ \epsilon = 10^{-8} \]
Test-bed study: Pöschl-Teller potential


Spectrum and Pseudospectrum of $L$ with $\log ||\text{Random}||_2 = -7$

\[
\tilde{V} = \tilde{V}_{\text{Pöschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1
\]

$\epsilon = 10^{-7}$
Test-bed study: Pöschl-Teller potential


Spectrum and Pseudospectrum of $L$ with $log||\text{Random}||_2 = -6$

\[ \tilde{V} = \tilde{V}_{\text{Pöschl–Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1 \]

\[ \epsilon = 10^{-6} \]
Test-bed study: Pöschl-Teller potential


Spectrum and Pseudospectrum of $L$ with $\log||\text{Random}||_2 = -5$

$$\tilde{V} = \tilde{V}_{\text{Poeschl–Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\epsilon = 10^{-5}$$
Test-bed study: Pöschl-Teller potential


Spectrum and Pseudospectrum of $L$ with $\log|| Random||_2 = -4$

\[ \tilde{V} = \tilde{V}_{\text{Poeschl–Teller}} + \epsilon E_{\text{Random}}, \quad || E_{\text{Random}} || = 1 \]

\[ \epsilon = 10^{-4} \]
Test-bed study: Pöschl-Teller potential


Spectrum and Pseudospectrum of $L$ with $\log||\text{Random}||_2 = -3$

$\tilde{V} = \tilde{V}_{\text{Poeschl–Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$

$\epsilon = 10^{-3}$
Test-bed study: Pöschl-Teller potential


Spectrum and Pseudospectrum of $L$ with $\log ||Random||_2 = -2$

\[
\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1
\]

$\epsilon = 10^{-2}$
Test-bed study: Pöschl-Teller potential


Spectrum and Pseudospectrum of $L$ with $\log ||\text{Random}||_2 = -1$

\[ \tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1 \]

$\epsilon = 10^{-1}$
Test-bed study: Pöschl-Teller potential


Spectrum and Pseudospectrum of $L$ with $\log ||\text{Random}||_2 = 0$

$$\tilde{V} = \tilde{V}_{\text{Poeschl–Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$\epsilon = 10^0$
From this we learn:

- **ALL overtones QNMs, unstable under high frequency perturbations:** instability grows as damping grows [JLJ, Macedo & Al Sheikh 21].

- Perturbations make **QNMs to “migrate” towards Pseudospectrum contour lines** (“extended pattern”, cf. Bauer-Fike theorem).

- **Slowest damped QNM, stable under high frequency perturbations:**
  - Directly from the Pseudospectrum.
  - From the size of the needed perturbations.

- It can be repeated with deterministic **high frequency** $k$ perturbations.

- For low frequency perturbations: much milder effect.

$$
\tilde{V} = \tilde{V}_{\text{Poeschl–Teller}} + \epsilon \cos(2\pi ky) , \quad k \gg 1
$$

- **It can also be seen:**
  - **Slowest damped QNM unstable under “infrared perturbations”** (cutting $V$ at large distances): Nollert’s instability of the fundamental QNM.
Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution $N$. Fixed random perturbation $\epsilon = 10^{-16}$.

Spectrum and Pseudospectrum of $L$ with $\log||Random||_2 = -50$

$$\tilde{V} = \tilde{V}_{\text{Pöschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$N = 120$
Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution $N$. Fixed random perturbation $\epsilon = 10^{-16}$.

Spectrum and Pseudospectrum of $L$ with $\log||\text{Random}||_2 = -50$

$$\tilde{V} = \tilde{V}_{\text{Pöschl–Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$N = 140$
Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution $N$. Fixed random perturbation $\epsilon = 10^{-16}$.

Spectrum and Pseudospectrum of $L$ with $\log||\text{Random}||_2 = -50$

\[ \tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1 \]
Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution $N$. Fixed random perturbation $\epsilon = 10^{-16}$.

Spectrum and Pseudospectrum of $L$ with $\log ||\text{Random}||_2 = -50$

\[ \tilde{V} = \tilde{V}_\text{Pöschl–Teller} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1 \]

$N = 160$
Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution $N$. Fixed random perturbation $\epsilon = 10^{-16}$.

Spectrum and Pseudospectrum of $L$ with $\log ||\text{Random}||_2 = -50$

\[ \tilde{\mathbb{V}} = \tilde{\mathbb{V}}_{\text{Pöschl-Teller}} + \epsilon |E_{\text{Random}}|, \quad ||E_{\text{Random}}|| = 1 \]

$N = 160$
Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution $N$. Fixed random perturbation $\epsilon = 10^{-16}$.

Spectrum and Pseudospectrum of $L$ with $\log \|\text{Random}\|_2 = -50$

$$\tilde{V} = \tilde{V}_{\text{Pöschl-Teller}} + \epsilon E_{\text{Random}}, \quad \|E_{\text{Random}}\| = 1$$

$N = 180$
Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution $N$. Fixed random perturbation $\epsilon = 10^{-16}$.

Spectrum and Pseudospectrum of $L$ with $\log||\text{Random}||_2 = -50$

\[ \tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1 \]

$N = 200$
From this we learn:

- Pseudospectrum contour lines seem to converge to QNM curves observed by Nollert [cf. Nollert 96].
- Similarity to Neutron-Star “w-modes”.

\[
\text{Re}(\omega_n) \quad \text{Im}(\omega_n)
\]

\[
-2 \times 10^1 \quad -1.5 \times 10^1 \quad -1 \times 10^1 \quad 0 \quad 5 \quad 10^1 \quad 1.5 \times 10^1 \quad 2 \times 10^1
\]
From this we learn:

- Pseudospectrum contour lines seem to converge to QNM curves observed by Nollert [cf. Nollert 96].
- Similarity to Neutron-Star “w-modes”.

Curvature/w-modes in neutron stars [Kokkotas & Schmidt 99]
Schwarzschild QNMs

Schwarzschild Pseudospectum: same qualitative behaviour (note: branch cut)

Highly damped QNMs unstable, slowest decaying QNMs stable.
Starting point: (scalar) wave equation in “tortoise” coordinates

On a stationary spacetime (with timelike Killing $\partial_t$):

$$\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V_\ell \right) \phi_{\ell m} = 0 ,$$

Dimensionless coordinates: $\bar{t} = t/\lambda$ and $\bar{x} = r_*/\lambda$ (and $\bar{V}_\ell = \lambda^2 V_\ell$).

Conformal hyperboloidal approach

$$\begin{cases} 
\bar{t} &= \tau - h(x) \\
\bar{x} &= f(x) 
\end{cases} .$$

- $h(x)$: implements the hyperboloidal slicing, i.e. $\tau = \text{const.}$ is a horizon-penetrating hyperboloidal slice $\Sigma_\tau$ intersecting future $\mathcal{I}^+$.
- $f(x)$: spatial compactification between $\bar{x} \in [-\infty, \infty]$ to $[a, b]$.
- Timelike Killing: $\lambda \partial_t = \partial_{\bar{t}} = \partial_{\tau}$.
First-order reduction: $\psi_{\ell m} = \partial_\tau \phi_{\ell m}$

$$\partial_\tau u_{\ell m} = i L u_{\ell m}, \quad \text{with} \quad u_{\ell m} = \begin{pmatrix} \phi_{\ell m} \\ \psi_{\ell m} \end{pmatrix}$$

where

$$L = \frac{1}{i} \begin{pmatrix} 0 & 1 \\ L_1 & L_2 \end{pmatrix}$$

$$L_1 = \frac{1}{w(x)} \left( \partial_x (p(x) \partial_x) - q(x) \right)$$

$$L_2 = \frac{1}{w(x)} \left( 2 \gamma(x) \partial_x + \partial_x \gamma(x) \right)$$

with

$$w(x) = \frac{f''^2 - h''^2}{|f'|} > 0, \quad p(x) = \frac{1}{|f'|}, \quad q(x) = |f'| V_\ell, \quad \gamma(x) = \frac{h'}{|f'|}.$$
Spectral problem

Taking Fourier transform, dropping \((l, m)\) (convention \(u(\tau, x) \sim u(x)e^{i\omega \tau}\)):

\[
Lu_n = \omega_n u_n. 
\]

where

\[
L = \frac{1}{i} \begin{pmatrix}
0 & 1 \\
L_1 & L_2
\end{pmatrix} 
\]

\[
L_1 = \frac{1}{w(x)} \left( \partial_x (p(x)\partial_x) - q(x) \right) \quad \text{(Sturm-Liouville operator)}
\]

\[
L_2 = \frac{1}{w(x)} \left( 2\gamma(x)\partial_x + \partial_x \gamma(x) \right)
\]

Conformal hyperboloidal approach: **No boundary conditions**

It holds \(p(a) = p(b) = 0\), \(L_1\) is “singular”: BCs “in-built” in \(L\).
Scalar product

Natural scalar product (where $\tilde{V}_\ell := q(x) > 0$):

$$\langle u_1, u_2 \rangle_E \equiv \frac{1}{2} \int_a^b \left( w(x) \bar{\psi}_1 \psi_2 + p(x) \partial_x \bar{\phi}_1 \partial_x \phi_2 + \tilde{V}_\ell \bar{\phi}_1 \phi_2 \right) dx ,$$

associated with the “total energy” of $\phi$ on $\Sigma_t$, defining the “energy norm”

$$\| u \|^2_E = \langle u, u \rangle_E = \int_{\Sigma_t} T_{ab}(\phi, \partial_\tau \phi) t^a n^b d\Sigma_\tau ,$$

Spectral problem of a non-selfadjoint operator

- Full operator $L$: not selfadjoint.
- $L_2$: dissipative term encoding the energy leaking at $\mathcal{I}^+$.
- $L$ selfadjoint in the non-dissipative $L_2 = 0$ case.

Non-normal operators spectral tools: “energy norm” for Pseudospectrum.
Conformal compactification

\[
\begin{align*}
\tilde{t} & = \tau - \frac{1}{2} \ln(1 - x^2) \\
\tilde{x} & = \arctanh(x)
\end{align*}
\iff
\begin{align*}
\tau & = \tilde{t} - \ln \cosh \tilde{x} \\
x & = \tanh \tilde{x}
\end{align*}
\]

mapping \([-\infty, \infty]\) to \([a, b] = [-1, 1]\).

Spectral problem

Operators in \(L\), with potential \(V(x) = V_0 \text{sech}^2(\tilde{x})\) (with \(V_0 = 1\)):

\[
\begin{align*}
L_1 & = \partial_x ((1 - x^2) \partial_x) - 1 \\
L_2 & = -(2x \partial_x + 1)
\end{align*}
\]

where:

\[
\begin{align*}
w(x) & = 1 \\
p(x) & = (1 - x^2) \\
q(x) & = \frac{V}{1 - x^2} =: \tilde{V}(x) \\
\gamma(x) & = -x.
\end{align*}
\]
Application to Schwarzschild

Schwarzschild potential

Axial (Regge-Wheeler) case (also for polar (Zerilli) parity):

\[ V_{\ell}^{s} = \left(1 - \frac{2M}{r}\right) \left( \frac{\ell(\ell + 1)}{r^2} + (1 - s^2) \frac{2M}{r^3} \right) , \]

where

- \( r_* = r + 2M \ln(r/2M - 1) \)
- \( s = 0, 1, 2 \), respectively, to the scalar, electromagnetic and gravitational cases.

Numerical Chebyshev methods: analyticity of \( V(x) \)

Bizoń-Mach coordinates used in Pöschl-Teller not well adapted now: the potential is non-analytic in \( x \), spoiling the accuracy of Chebyshev’s methods.

Same problem for the polar (Zerilli) case.

Solution

We resort rather to the 'minimal gauge' hyperboloidal slicing [Ansorg, Macedo 16; Macedo 18] guaranteeing the analyticity of the Schwarzschild potential.
Conformal compactification: “minimal gauge” [Ansorg, Macedo 16; Macedo 18]

\[
\begin{align*}
\tilde{t} &= \tau - \frac{1}{2} \left( \ln \sigma + \ln(1 - \sigma) - \frac{1}{\sigma} \right) \\
\tilde{x} &= \frac{1}{2} \left( \frac{1}{\sigma} + \ln(1 - \sigma) - \ln \sigma \right),
\end{align*}
\]

mapping \([-\infty, \infty]\) to \([a, b] = [0, 1]\) (we use \(\sigma\), rather than \(x\)).

Spectral problem

Operators in \(L\):

\[
L_1 = \frac{1}{1 + \sigma} \left[ \partial_\sigma \left( \sigma^2 (1 - \sigma) \partial_\sigma \right) - \left( \ell (\ell + 1) + (1 - s^2) \sigma \right) \right]
\]

\[
L_2 = \frac{1}{1 + \sigma} \left( (1 - 2\sigma^2) \partial_\sigma - 2\sigma \right).
\]

where:

\[
w(\sigma) = 1 + \sigma, \quad p(\sigma) = \sigma^2 (1 - \sigma), \quad q(\sigma) = \frac{V}{\sigma^2 (1 - \sigma)} =: \tilde{V}(\sigma), \quad \gamma(\sigma) = 1 - 2\sigma^2.
\]
Trying to understand: the “current picture”... in pictures

Pöschl-Teller, $L_2 = 0$:
Trying to understand: the “current picture”... in pictures

Pöschl-Teller, $L_2 \neq 0$:

$\log \left( \frac{\kappa_n}{\kappa_0} \right)$

$|\delta V_r| = 10^{-16} \times$ $|\delta V_r| = 10^{-8}$ $+$ $|\delta V_r| = 10^0$ $\ast$

$|\delta V_d| = 10^{-8}, k = 1$ $\square$ $|\delta V_d| = 10^{-8}, k = 10$ $\triangle$ $|\delta V_d| = 10^{-8}, k = 20$ $\diamond$
Trying to understand: the “current picture”... in pictures

Schwarzschild ($s = 2$, $\ell = 2$):

- $\log \left( \frac{\kappa_n}{\kappa_0} \right)$
- $\Re(\omega_n)$
- $\Im(\omega_n)$

- $||\delta V_d|| = 10^{-8}$, $k = 1$
- $||\delta V_d|| = 10^{-8}$, $k = 20$
- $||\delta V_d|| = 10^{-8}$, $k = 60$
- $||\delta V_d|| = 10^{-4}$, $k = 20$
Comparison Pöschl-Teller versus Schwarzschild:

Remarks:
- High frequency perturbations: random $\delta V_r$, deterministic $\delta V_d \sim \cos(2\pi k x)$.
- Fundamental QNM stable.
- Perturbed QNMs “migrate” towards $\epsilon$—contour lines of Pseudospectra.
- 'Universality' phenomenon?
Trying to understand: the “current picture”... in pictures

Black Hole and Neutron Star QNMs

Comparison with:
- Nollert’s high-frequency Schwarzschild perturbations.
- Nollert’s remark on Neutron Stars (w-modes) curvature QNMs.
Trying to understand: the “current picture”... in pictures

“Duality” QNMs (long-range potentials) and Regge poles (compact support potentials)?

QNM of an spherical obstacle [Stefanov 06]:

- Red-diamonds: fixed “$n$”, running angular $\ell$.
- Blue-diamonds: fixed $\ell$ (here $\ell = 20$), running $n$. 
The Problem in a nutshell: (asymptotically flat) BH QNM instability

Non-normal operators: spectral instability and Pseudospectrum

Hyperboloidal approach to QNMs: the Pöschl-Teller toy model

Black Hole QNM ultraviolet instability

Discussion, Conclusions and Perspectives: a low-regularity problem
Connecting with Lax-Phillips: Keldysh expansion

Hilbert space: Keldysh’s expansion of the resolvent \([Keldysh 51, 71]\)

Given \(v_j\) and \(w_j\) (right- . left and right eigenvectors of \(L\)), with \(\langle w_n, v_n \rangle = -1\), the resolvent of \(L\) can be expanded in a domain \(\Omega\) around the poles as

\[
(L - \lambda)^{-1} = \sum_{\lambda_j \in \Omega} \frac{|v_j\rangle \langle w_j|}{\lambda - \lambda_j} + H(\lambda), \quad \lambda \in \Omega \setminus \sigma(L)
\]

QNM resonant expansions, “1st-order time reduction” \([Al Sheikh, JLJ, Gasperin 21, 22]\)

The field \(u\) satisfying \(\partial_\tau u = iL u\), with \(u(\tau = 0, x) = u_0\), can be written in an “asymptotic expansion” as

\[
u(\tau, x) = \sum_{j=1}^{N} e^{i\omega_j \tau} \kappa_j \langle \hat{w}_j | u_0 \rangle_E \hat{v}_j + E_N(\tau; S)
\]

with

\[
\|E_N(\tau; S)\|_E \leq C_N(a, L)e^{-a\tau}\|S\|_E
\]

Note the presence of the condition number \(\kappa_j\).
Connecting with Lax-Phillips: Keldysh expansion

Hilbert space: Keldysh’s expansion of the resolvent [Keldysh 51, 71]

Given $v_j$ and $w_j$ (right- . left and right eigenvectors of $L$), with $\langle w_n, v_n \rangle = -1$, the resolvent of $L$ can be expanded in a domain $\Omega$ around the poles as

$$(L - \lambda)^{-1} = \sum_{\lambda_j \in \Omega} \frac{|v_j\rangle\langle w_j|}{\lambda - \lambda_j} + H(\lambda), \quad \lambda \in \Omega \setminus \sigma(L)$$

QNM resonant expansions, “scattered field” [Al Sheikh, JLJ, Gasperin 21, 22]

In terms of the scattered field satisfying the 2nd order equation

$$\phi(\tau, x) \sim \sum_n e^{i\omega_j \tau} a_n \hat{\phi}_j^R(x), \quad \phi(t, x) \sim \sum_n e^{i\omega_j t} a_n e^{i\omega_j h(x)} \hat{\phi}_j^R(x)$$

with $$a_j = \frac{\kappa_j}{2} \left( \langle \hat{\phi}_j^L, \varphi_0 \rangle_{H^1(V,p)} - i\omega_j \langle \hat{\phi}_j^L, \varphi_1 \rangle_{(2,w)} \right)$$

that recovers the expression in terms of normal modes in the selfadjoint (normal) case: $\kappa_j = 1$, $\hat{\phi}_j^R(x) = \hat{\phi}_j^L(x)$. 
Extracting QNM: Mode Filtering

\[ \Phi_{\text{spec}}^N(t) := \sum_{n=0}^{N} A_n e^{i\omega_n t} \]

\[ F^N(t) = \Phi_{\text{evol}}(t) - \Phi_{\text{spec}}^N(t) \]
Are the perturbed QNMs in the ringdown signal? Yes

Extracting QNM: Mode Filtering

\[ \Phi_{\text{spec}}^N(t) := \sum_{n=0}^{N} A_n e^{i\omega_n t} \]

\[ \mathcal{F}^N(t) = \Phi_{\text{evol}}(t) - \Phi_{\text{spec}}^N(t) \]
Extracting QNM: Mode Filtering

\[ \Phi^N_{\text{spec}}(t) := \sum_{n=0}^{N} A_n e^{i \omega_n t} \]

\[ \mathcal{F}^N(t) = \Phi_{\text{evol}}(t) - \Phi^N_{\text{spec}}(t) \]

modes with \( n = 0, 1 \) removed

modes with \( n = 0, 1, 3, 4, 5, 6, 7 \) removed
Emerging picture: "Universality" of compact-object QNMs

Logarithm QNM-free regions: Pseudospectrum contour lines [JLJ, Macedo, Al Sheikh 21]

- QNM overtones: ultraviolet (high-frequency) unstable.
- Fundamental QNM: ultraviolet (high-frequency) stable.
  (Warning: "Flea on the Elephant" effect for "trapping potentials" [Jona-Lasinio et al. 81, 84; Helffer & Sjöstrand 83, 85; Simon 85; Cheung et al. 21,...])
- Logarithmic $|\text{Re}(\omega)| \gg 1$ asymptotics: $\text{Im}(\omega) \sim C_1 + C_2 \ln(|\text{Re}(\omega)| + C_3)$
Emerging picture: "Universality" of compact-object QNMs

$C^p$ regularity: Regge QNM branches, logarithmic asymptotics

- Compact support potential: **theorem** by Zworski (Weyl law) [Zworski 87] (Note: this **fully** explains Nollert's results [Nollert 96]
- Non-compact support, $C^p<\infty$ potential: WKB Berry's treatment [Berry 82].
- Related results by Nollert-Price [Nollert & Price 99]
- Polytropic neutron stars: $w$-modes [Zhang, Wu & Leung 11]

\[
\begin{align*}
\text{Re}(\omega_n) & \sim \pm \left( \frac{\pi}{L} n + \frac{\pi \gamma_p}{2L} \right) \\
\text{Im}(\omega_n) & \sim \frac{1}{L} \left[ \gamma \ln \left( \left( \frac{\pi}{L} n + \frac{\pi \gamma_p}{2L} \right) + \frac{\pi \gamma \Delta}{2L} \right) - \ln S \right]
\end{align*}
\]

- "Length scale" $L$
- "Small structure" coefficient $\gamma$
- "Strength coefficient" $S$
- "Real shift" $\gamma \Delta$
- "Parity term" $\gamma_p$
Emerging picture: "Universality" of compact-object QNMs

Regular perturbations: “Nollert-Price-like” branches and “inner QNMs”

- Always above the logarithm pseudospectra lines: “Lamis’ Graal”.
- “Nollert-Price-like”: different branches, transitions at critical values.
- “Inner modes”: appearing at the critical values. Weyl law.
- Nollert-Price branches towards logarithm pseudospectra as perturbation frequency increases: Do they get to the logarithmic pseudospectra lines?
Discussion, Conclusions and Perspectives: a low-regularity problem

From Burnett conjecture to a Regge QNM conjecture

Burnett’s conjecture: high-frequency limit of spacetimes [Burnett 89; Huneau, Luk 18, 19]

The limit of high-frequency oscillations of a vacuum spacetime is effectively described by an effective matter spacetime described by massless Einstein-Vlasov.

Luk & Rodnianski: “null shells” as low-regularity problems in Einstein equations

- Spikes are more efficient in triggering instabilities [Gasperin & JLJ 21]:
  \[ |\delta g_{ab}| ||\delta L||_E \sim \max_{\text{supp}(\delta g)} |\partial \delta g_{ab}|^2 \sim \max_{\text{supp}(\delta g)} \rho_E(\delta g; x) \]

- Luk & Rodnianski’s treatment [Luk & Rodnianski 20]: Allowing for “concentrations”, the infinite high-frequency limit is a low regularity limit
  \[ g_n \to g_\infty \text{ in } C^0 \cap H^1 , \quad \partial g_n \to \partial g_\infty \text{ in } L^2 \]

Regge QNM branches conjecture: a low-regularity problem [JLJ, Macedo, AL Sheikh 21, Gasperin & JLJ 21]

In the limit of infinite frequency, generic ultraviolet perturbations push BH QNMs in Nollert-Price branches to asymptotically logarithmic branches along the QNM-free region boundary, exactly following the Regge QNM asymptotic pattern.
QNMs and norms: “Definition” versus “Stability” problems

**Definition of QNMs:** we can (need to) choose the norm to control QNMs

- Ansorg & Macedo [Ansorg & Macedo 16]: if $C^\infty$-regularity, then every point in the upper complex plane is an eigenvalue. Need of “more control”.


Hilbert spaces of $(\sigma, 2)$-Gevrey functions on $[0, \rho_0]$: $f, g \in C^\infty((0, \rho_0); \mathbb{C})$

$$\langle f, g \rangle_{G^2_{\sigma, 1, \rho_0}} = \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{n!^2(n+1)!^2} \int_{0}^{\rho_0} \partial^n f \partial^n \bar{g} d\rho$$

**Stability of QNMs:** we cannot choose the norm, the ”physics” chooses for us

The system chooses what is a ”big and small” perturbation: here, the energy.

$$\| (\phi, \psi) \|_E = E = \int_\Sigma T_{ab} t^a n^b d\Sigma \sim \| \phi \|_{H^1_V} + \| \psi \|_{L^2}$$

“Energy-Pseudospectrum = Chart of the Gevrey Ocean”
A small high-frequency perturbation in the energy, is a huge one in Gevrey terms.

- For a given $V$, QNM eigenfunctions are defined to be in class Gevrey-2.
- Small (in the energy norm) perturbations $\delta V$ of $V$ lead to very different QNM $\omega_n$'s, respective eigenfunctions being indeed Gevrey-2.
Conclusions and Perspectives

Conclusions

- Numerical evidence of instability of QNM overtones under high-frequency perturbations in the effective potential.
- Independent support from Pseudospectrum of non-perturbed operator.
- Hints towards **Universality of compact-object QNM branches** in the infinite high-frequency limit: a low regularity problem.

Perspectives

- **Can we measure the regularity of spacetime?**
  [paraphrasing: “What is the regularity of spacetime?” [question from Bruce Allen]]
- To transform presented heuristic and numerical evidences into Theorems:
  i) Semiclassical estimates of the pseudospectrum.
  ii) Regge QNM conjecture from Burnett’s conjecture.
- Implement Pseudospectrum in Gevrey-2 scalar products: trivialization?
- Study stability of the time domain signal under high-frequency perturbations.
- ...

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Discussion, Conclusions and Perspectives: a low-regularity problem
Conclusions and Perspectives

The “Gevrey Ocean” and the Dijon Legacy...
Conclusions and Perspectives

“Truth suffers from much Analysis”
(Ancient Fremen saying) Dune Messiah, Frank Herbert

A 'hint of Geometry' perhaps...?