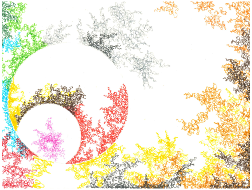
On the stability of black hole quasi-normal modes: a pseudospectrum approach

José Luis Jaramillo

(based on joint work with Rodrigo P. Macedo, Lamis Al Sheikh and Edgar Gasperín)

Institut de Mathématiques de Bourgogne (IMB)
Université de Bourgogne Franche-Comté
Jose-Luis.Jaramillo@u-bourgogne.fr



Resonances, Inverse Problems and Seismic Waves 2021

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Scheme

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Black Hole QNM instabilities [cf. Nollert 96, Nollert & Price 99]

Perturbation theory on a Schwarzschild Black Hole: spherically symmetric case

Scalar, electromagnetic and gravitational perturbations reduced to (Minkowski) 1+1 wave equation for $\phi_{\ell m}(t, r_*)$ with a potential V_ℓ [Regge-Wheeler 57, Zerilli 70]:

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V_\ell\right)\phi_{\ell m} = 0 \quad , \quad t \in]-\infty, \infty[\ , \ r^* \in]-\infty, \infty[$$

Schwarzschild quasi-normal modes

Convention for a "mode": $\phi_{\ell m}(t, r_*) \sim e^{i\omega t}\hat{\phi}_{\ell m}(r_*)$.
"Spectral" problem with **"outgoing boundary" conditions**:

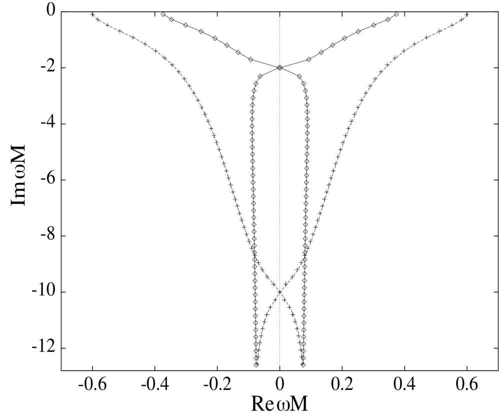
$$\left(-\frac{\partial^2}{\partial r_*^2} + V_\ell\right)\hat{\phi}_{\ell m} = \omega^2\hat{\phi}_{\ell m} \quad , \quad r^* \in]-\infty, \infty[$$

$$\hat{\phi}_{\ell m} \sim e^{-i\omega r_*}\ ,\ (r_* \to \infty) \quad , \quad \hat{\phi}_{\ell m} \sim e^{i\omega r_*}\ ,\ (r_* \to -\infty)$$

Time evolution stability: $\mathrm{Im}(\omega) > 0$. Exponential divergence of $\hat{\phi}_{\ell m}$ at $\pm\infty$.

Black Hole QNM instabilities [cf. Nollert 96, Nollert & Price 99]

Schwarzschild gravitational QNMs

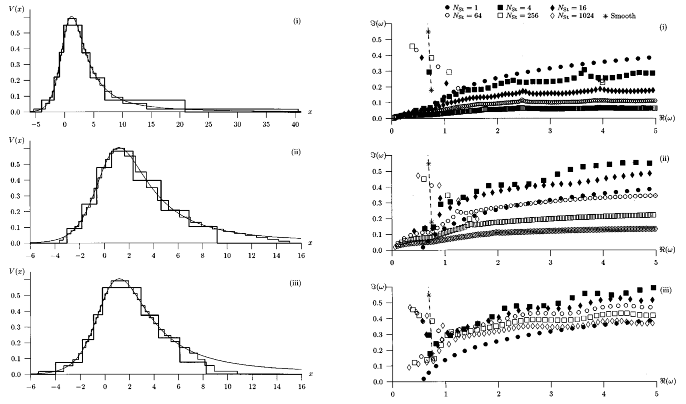


Schwarzschild QNMs ($\ell = 2$ diamonds, $\ell = 3$ crosses) [e.g. Kokkotas & Schmidt 99]

QNM frequencies ω_n are invariant probes into the background geometry

Black Hole QNM instabilities [cf. Nollert 96, Nollert & Price 99]

Nollert's work on stair-case discretizations of Schwarzschild (revisited in [Daghigh, Green & Morey 20, arXiv:2002.0725])



- Instability of the slowest decaying QNM (but ringdown "stability").
- Instabilities of "highly damped QNMs".

Posing the problem

Consider the operator on functions (defined on "appropriate" functional spaces) with non-compact-domain and with $V > 0$ with appropriate decay at infinity:

$$P_V = -\Delta + V$$

QNMs in the theory of Scattering Resonances

[Lax & Phillips, Vainberg; Sjöstrand, Zworski, Petkov, Iantchenko, ...many others; e.g. Dyatlov & Zworski 20]

Resolvent $R_V(\lambda) = (P_V - \lambda)^{-1}$ analytic $\mathrm{Im}(\lambda) > 0$. Scattering resonances: poles of the meromorphic extension of (truncated) $R_V(\lambda)$ to $\mathrm{Im}(\lambda) < 0$.

QNMs as a "proper" eigenvalue problem: **non-selfadjoint operators**

- "Complex scaling" [Simon 78, Reed & Simon 78, Sjöstrand...]: not the approach followed here.
- Hyperboloidal approach [Friedman & Schutz 75, Schmidt 93, Bizon, Zenginoglu 11, Vasy 13, Warnick 15, Ansorg & P.-Macedo 16, Gajic & Warnick 19, Bizon et al. 20, Galkowski & Zworski 20, ...]
  Problem in terms of "eingenvalue problem" of **non-selfadjoint operator** L:

$$\boxed{L\, u_{\ell m} = \omega\, u_{\ell m}} \quad , \quad u_{\ell m} \in H \text{ (Hilbert space)}$$

  - **Geometric boundary conditions**: Null infinity reached by hyperboloidal slices.
  - **Regularity conditions on** $u_{\ell m}$: choice of appropriate H, then $\omega \in \sigma_p(L)$.

Scheme

Spectral Theorem. Normal and 'non-normal' operators

Normal operators: Spectral Theorem

- **Normality**: denoting the adjoint matrix by $L^\dagger$, then L is normal iff

$$[L, L^\dagger] = LL^\dagger - L^\dagger L = 0$$

  Matrix examples: symmetric, hermitian, orthogonal, unitary...

- **Spectral Theorem** ("moral statement"):
  L is normal iff is unitarily diagonalisable.

Note: this depends on the adjoint $L^\dagger$, then on the Hilbert space (scalar product).

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'Non-normal' operators, $[L, L^\dagger] \neq 0$: no Spectral Theorem

- Completeness more difficult to study.
- Eigenvectors not necessarily orthogonal.
- **Spectral instabilities**.

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Methodology here: "exploration" stage

Numerical spectral methods: **Chebyshev polynomials** truncations.

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Example of spectral instability

$$L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c \quad , \quad a, b, c \in \mathbb{R}$$

acting on functions in $L^2([0,1])$, with homogeneous Dirichlet conditions (Chebyshev finite-dimensional matrix approximates).

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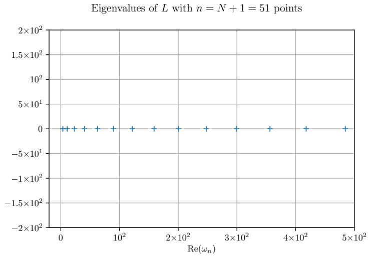

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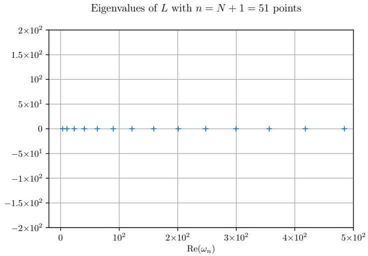

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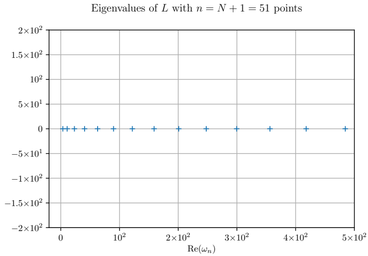

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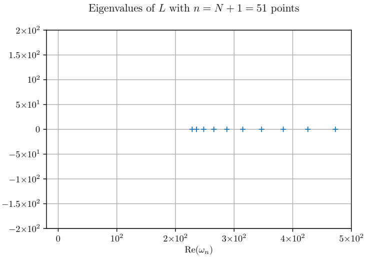

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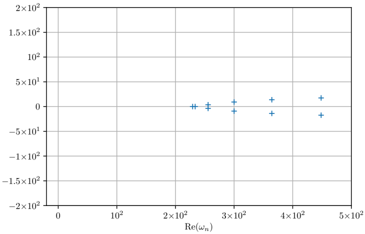
Eigenvalues of L with $n = N + 1 = 51$ points



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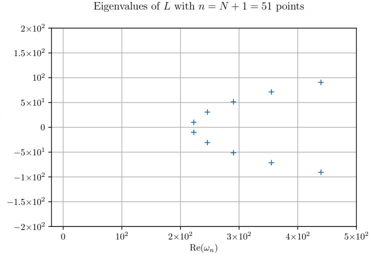

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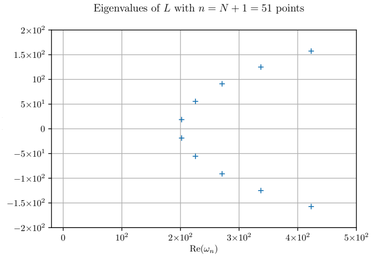

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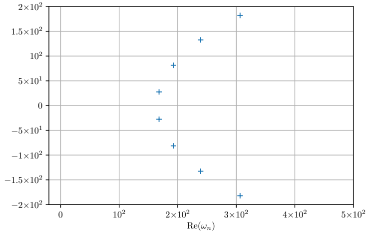

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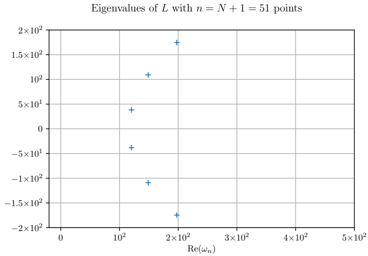

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Spectral (in)stability: eigenvalue condition number

Left- and right-eigenvectors, respectively u_i and v_i, of A

$$A^\dagger u_i = \bar{\lambda}_i u_i \ \ (\Leftrightarrow u_i^\dagger A = \lambda_i u_i^\dagger) \ \ , \ \ A v_i = \lambda_i v_i \ \ , \ \ i \in \{1, \ldots, n\} \ ,$$

Perturbation theory of eigenvalues [cf. Kato 80, ...; e.g. Trefethen, Embree 05]:

$$A(\epsilon) \ = \ A + \epsilon \delta A \ \ , \qquad ||\delta A|| = 1 \ .$$

$$|\lambda_i(\epsilon) - \lambda_i| = \epsilon \frac{|\langle u_i, \delta A \, v_i(\epsilon)\rangle|}{|\langle u_i, v_i\rangle|} \leq \epsilon \frac{||u_i|| \, ||\delta A \, v_i||}{|\langle u_i, v_i\rangle|} + O(\epsilon^2) \leq \epsilon \frac{||u_i|| \, ||v_i||}{|\langle u_i, v_i\rangle|} + O(\epsilon^2).$$

Eigenvalue condition number: $\kappa(\lambda_i)$

$$\boxed{\kappa(\lambda_i) = \frac{||u_i||||v_i||}{|\langle u_i, v_i\rangle|}}$$

Spectral (in)stability and Pseudospectrum

Pseudospectrum

Given $\epsilon > 0$, the ϵ-pseudospectrum $\sigma_\epsilon(L)$ of L is defined as [e.g Trefethen & Embree 05]:

$$
\begin{aligned}
\sigma_\epsilon(L) &= \boxed{\{\lambda \in \mathbb{C}, \text{ such that } \lambda \in \sigma(L + \delta L) \text{ for some } \delta L \text{ with } ||\delta L|| < \epsilon\}} \\
&= \{\lambda \in \mathbb{C}, \text{ such that } ||Lv - \lambda v|| < \epsilon \text{ for some } v \text{ with } ||v|| = 1\} \\
&= \{\lambda \in \mathbb{C}, \text{such that } ||(\lambda I - L)^{-1}|| > \epsilon^{-1}\}
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Normal case: bounds on the norm of the resolvent $R_L(\lambda) = (\lambda I - L)^{-1}$

Given $\lambda \in \mathbb{C}$ and $\sigma(L)$ the spectrum of L, it holds

$$||(\lambda I - L)^{-1}||_2 = \frac{1}{\mathrm{dist}(\lambda, \sigma(L))}$$

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Non-normal case: bad control on the resolvent $R_L(\lambda)$. **Pseudospectrum**

The norm of the resolvent can become very large far from the spectrum:

$$||(\lambda I - L)^{-1}||_2 \leq \frac{\kappa}{\text{dist}(\lambda, \sigma(L))}$$

where κ is a "condition number" assessing the lack of proportionality of 'left' and 'right' eigenvectors of L, and can become very large in the non-normal case.

Bauer-Fike theorem. Random perturbations

Pseudospectrum and condition number: Bauer-Fike theorem [e.g Trefethen & Embree 05]

Defining "tubular neighbourhood" of radius ϵ around $\sigma(A)$

$$\Delta_\epsilon(A) = \{\lambda \in \mathbb{C} : \mathrm{dist}\,(\lambda, \sigma(A)) < \epsilon\} \ ,$$

it holds: $\Delta_\epsilon(A) \subseteq \sigma_\epsilon(A)$. For normal operators: $\sigma_\epsilon(A) = \Delta_\epsilon(A)$.
Non-normal case, $\kappa(\lambda_i) \neq 1$, it holds (for small ϵ):

$$\sigma_\epsilon(A) \subseteq \bigcup_{\lambda_i \in \sigma(A)} \Delta_{\epsilon\kappa(\lambda_i)+O(\epsilon^2)}(\{\lambda_i\}) \ ,$$

Therefore $\sigma_\epsilon(A)$ larger tubular neighbourhood of radius $\sim \epsilon\kappa(\lambda_i)$.

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Random perturbations and Pseudospectrum

Random perturbations ΔL with $||\delta L|| < \epsilon$ "push" eigenvalues into $\sigma_\epsilon(A)$, providing an insightful and systematic manner of exploring $\sigma_\epsilon(L)$.

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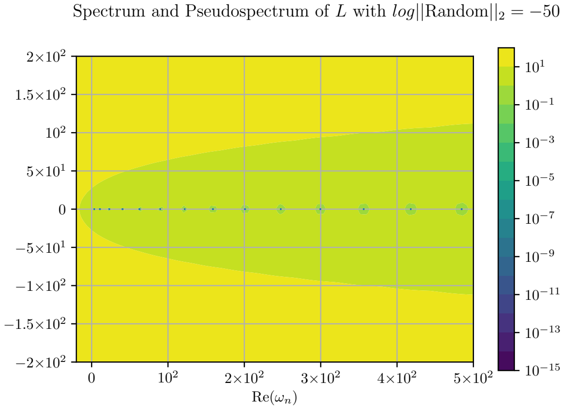
The 'role' of random perturbations [Sjöstrand 19; Hager 05, Montrieux, Nonnenmacher, Vogel,...]

Random perturbations improve the analytical behaviour of $R_L(\lambda)$!!!

Spectral (in)stability and Pseudospectrum: illustration

Pseudospectrum of: $L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c + \epsilon E_{\text{Random}}$



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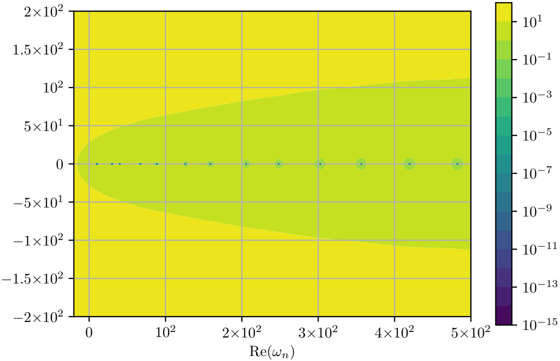
Spectrum and Pseudospectrum of L with $log||\text{Random}||_2 = 1$



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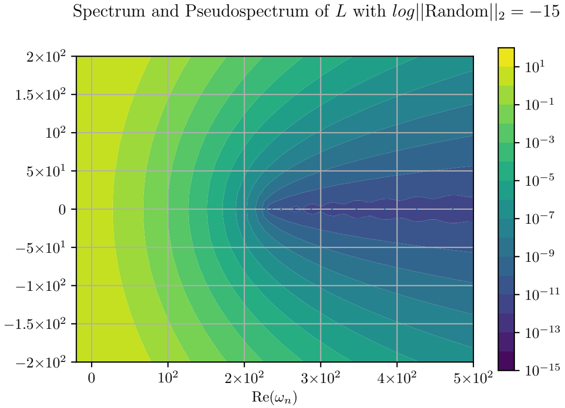

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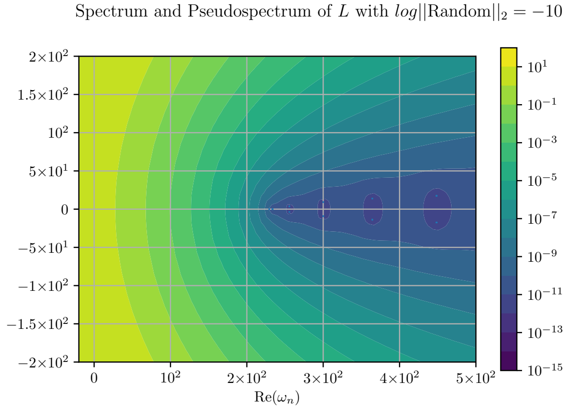

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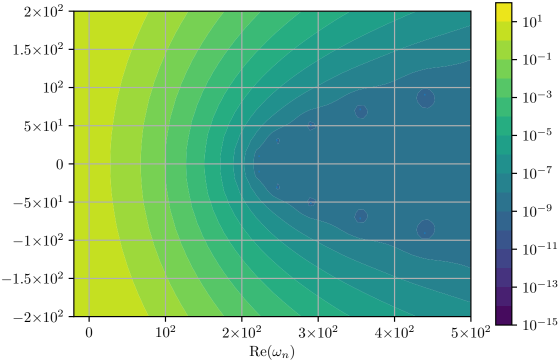
Spectrum and Pseudospectrum of L with $log||\mathrm{Random}||_2 = -8$



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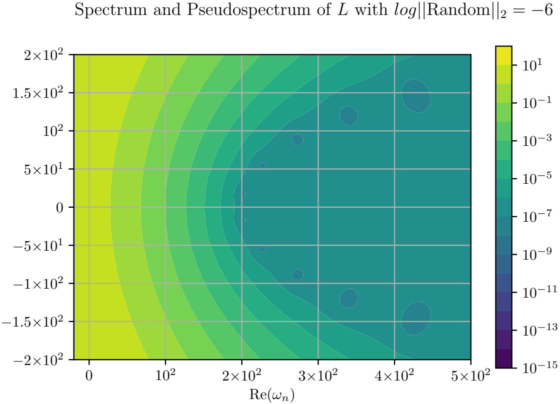

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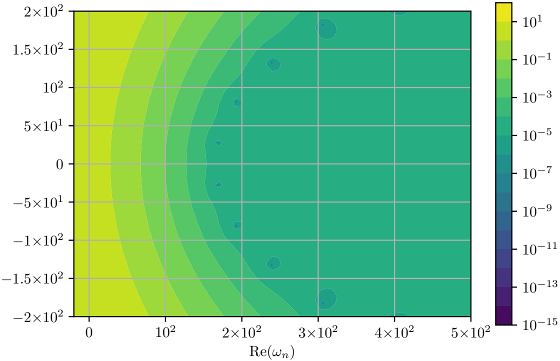
Spectral (in)stability and Pseudospectrum: illustration

Pseudospectrum of: $L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c + \epsilon E_{\text{Random}}$

Spectrum and Pseudospectrum of L with $log||\text{Random}||_2 = -4$



$a = -1, \; b = 30, \; c = 1, \; \epsilon = 10^{-4}$

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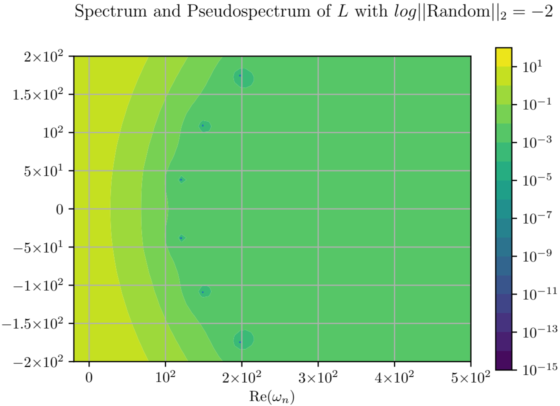

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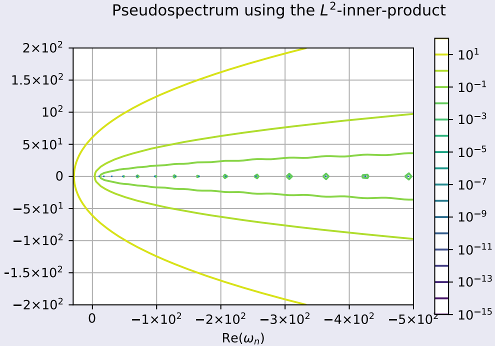
The relevance of the scalar product: assessing large/small

The illustrative operator: $L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c \quad , \quad a, b, c \in \mathbb{R}$

- Non-selfadjoint in standard $L^2([0,1])$ for $b \neq 0$.
- Formally normal!
- Non-normal: domain of $L^\dagger L$ and $LL^\dagger$ different.
- But actually self-adjoint...

Cast in Sturm-Liouville form: selfadjoint for appropriate scalar product $\langle\cdot,\cdot\rangle_w$!!!



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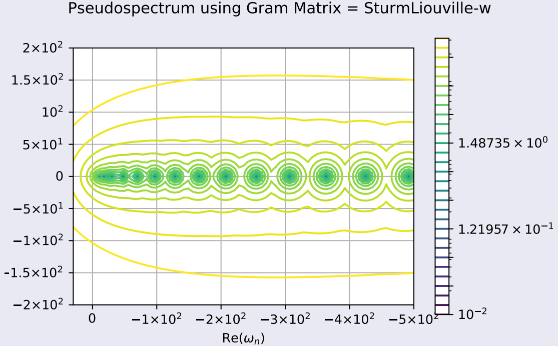
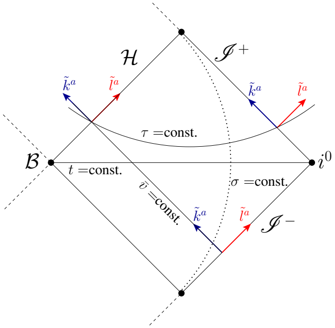


Scheme

Hyperboloidal slices: geometric outgoing BCs at $\mathscr{I}^+$

Hyperboloidal approach to QNMs

- **Spectral problem**: homogeneous wave equation with purely outgoing boundary conditions.
- Outgoing BCs naturally imposed at $\mathscr{I}^+$.
- Outgoing BCs actually "incorporated" at $\mathscr{I}^+$:
  - Geometrically: null cones outgoing.
  - Analytically: BCs encoded into a singular operator, **"BCs as regularity conditions"**.
- **Eigenfunctions** do not diverge when $x \to \infty$: actually **integrable**. Key to Hilbert space.



Hyperboloidal slices: geometric outgoing BCs at $\mathscr{I}^+$

Hyperboloidal approach to QNMs

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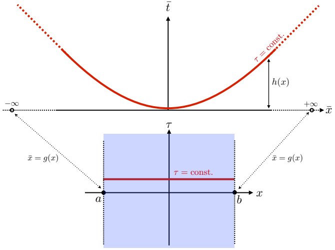

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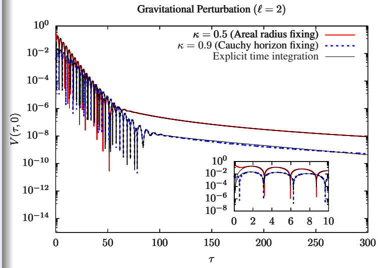

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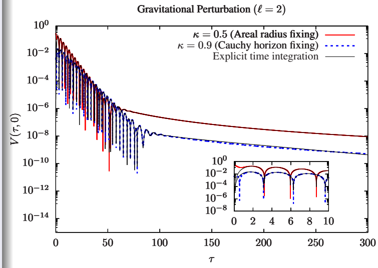

1. Wave problem in spherically symmetric asymptotically flat case

As starting point, consider the problem for a $\phi_{\ell m}$ mode in tortoise coordinates:

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V_\ell\right)\phi_{\ell m} = 0 \quad , \quad t \in]-\infty, \infty[\; , \; r^* \in]-\infty, \infty[$$

Compactification along hyperboloidal slices

2. Choice of hyperboloidal foliation and compactification

Make the change to Bizoń-Mach variables [Bizoń & Mach 17]:

$$\left\{ \begin{array}{lll} \tau & = & t - \ln\left(\cosh r_*\right) \\ x & = & \tanh r_* \end{array} \right. \quad , \quad \tau \in]-\infty, \infty[\; , \; x \in]-1, 1[$$

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3. Wave equation in hyperboloidal coordinates: no boundary conditions allowed

For $x = \pm 1$, $V_\ell = 0$. In the interior, $x \in]-1, 1[$:

$$\left(\partial^2_\tau + 2x\partial_\tau\partial_x + \partial_\tau + 2x\partial_x - (1-x^2)\partial^2_x + \tilde{V}_\ell\right)\phi_{\ell m} = 0 \ ,$$

with $\tilde{V}_\ell = \dfrac{V_\ell}{(1-x^2)}$.

Wave equation: reduction to first order system

4. Evolution equation in first order form

Introducing the auxiliary field

$$\psi_{\ell m} = \partial_\tau \phi_{\ell m} \ ,$$

we can write the wave equation in first-order form:

$$\partial_\tau \begin{pmatrix} \phi_{\ell m} \\ \psi_{\ell m} \end{pmatrix} = \left(\begin{array}{c|c} 0 & 1 \\ \hline (1-x^2)\partial_x^2 - 2x\partial_x - \tilde{V}_{\ell m} & -(2x\partial_y + 1) \end{array} \right) \begin{pmatrix} \phi_{\ell m} \\ \psi_{\ell m} \end{pmatrix} \ .$$

Spectral problem: first order formulation

5. Eigenvalue problem for a non-selfadjoint operator, no BCs

Our problem: study

$$L \begin{pmatrix} \phi_{\ell m} \\ \psi_{\ell m} \end{pmatrix} = \omega \begin{pmatrix} \phi_{\ell m} \\ \psi_{\ell m} \end{pmatrix} \quad , \quad L = \left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array} \right) ,$$

where

$$L_1 = (1 - x^2)\partial_x^2 - 2x\partial_x - \tilde{V}_{\ell m} \quad , \quad L_2 = -(2x\partial_x + 1) \ .$$

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QNM problem as a “proper” eigenvalue problem. But...**Hilbert space?**

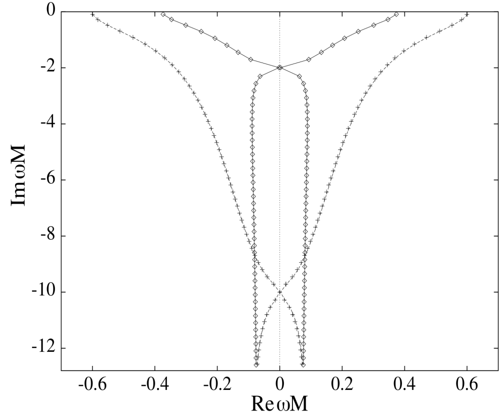
Spectral problem in a Hilbert space with **“Energy” scalar product** ($\tilde{V} > 0$):

$$\left\langle \begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix}, \begin{pmatrix}\phi_2\\ \psi_2\end{pmatrix} \right\rangle_E = \int_{\Sigma_\tau} \left(\bar{\psi}_1\psi_2 + (1-x^2)\partial_x\bar{\phi}_1\partial_x\phi_2 + \tilde{V}\bar{\phi}_1\phi_2\right) d\Sigma_t$$

Scheme

Black Hole QNM instabilities [cf. Nollert 96, Nollert & Price 99]

Schwarzschild gravitational QNMs



Schwarzschild QNMs ($\ell = 2$ diamonds, $\ell = 3$ crosses) [e.g. Kokkotas & Schmidt 99]

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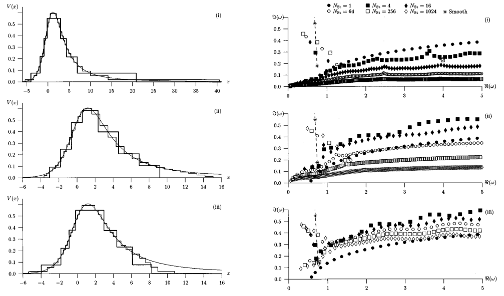
Nollert's work on stair-case discretizations of Schwarzschild (revisited in [Daghigh, Green & Morey 20, arXiv:2002.0725])



- Instability of the slowest decaying QNM (but ringdowm "stability").
- Instabilities of "highly damped QNMs".
- Various interests in BH QNM perturbations:
  - i) "Dirty" asrophysical black holes [Leung et al. 97; Barausse, Cardoso & Pani 14;...]
  - ii) Quantum (highly damped QNMs/high frequency instability) [Hod 98; Maggiore 08; Babb, Daghigh & Kunstatter 11; Ciric, Konjik & Samsarov 19, Olmedo; ...].

Test-bed study: Pöschl-Teller potential

Pöschl-Teller potential [JLJ, Macedo & Al Sheikh 21] (toy-model in [Bizoń, Chmaj & Mach 20])

$$V = V_o \,\mathrm{sech}^2(r_*) = V_o(1-y^2) \quad \Longrightarrow \quad \boxed{\tilde{V} = V_o}$$

Particularly simple form (scalar field in de Sitter, $m^2 = V_o$ [Bizoń, Chmaj & Mach 20])

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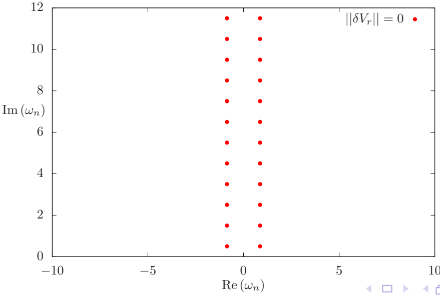
Pölsch-Teller QNM Perturbed-Spectra: Random Potential



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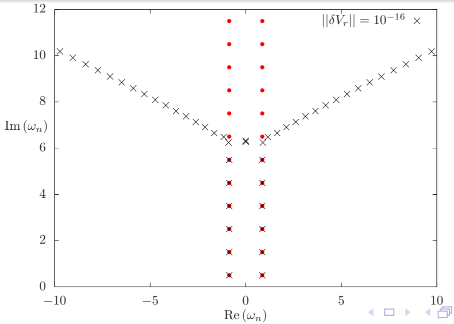

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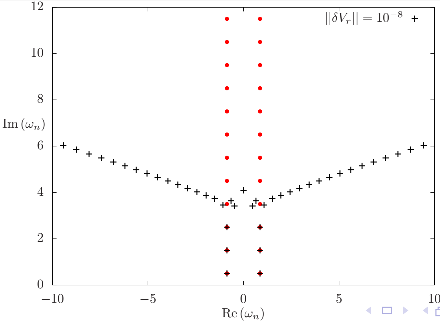

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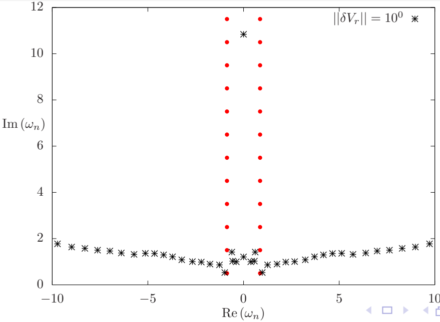

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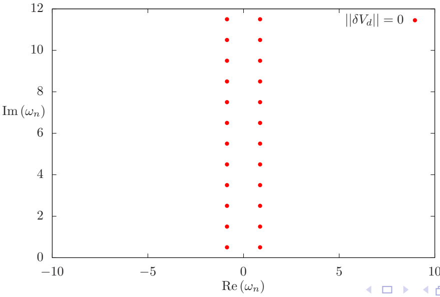
Pölsch-Teller QNM Perturbed-Spectra: Deterministic Potential



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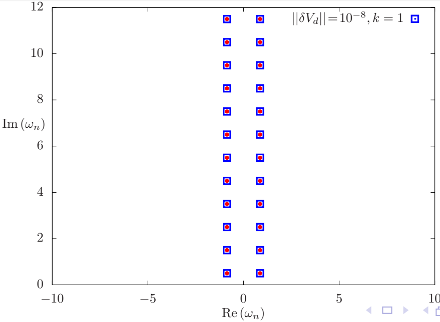

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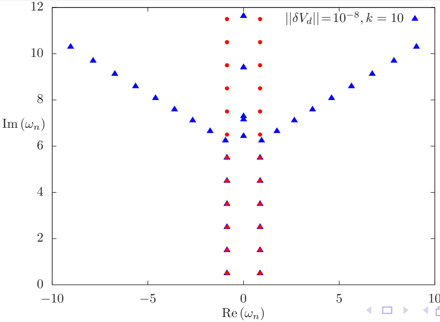

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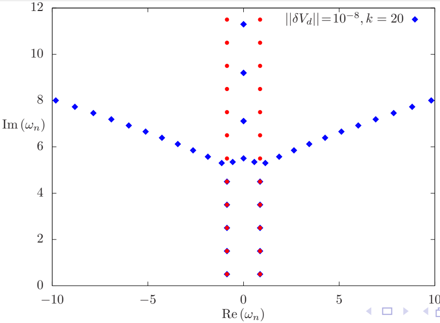

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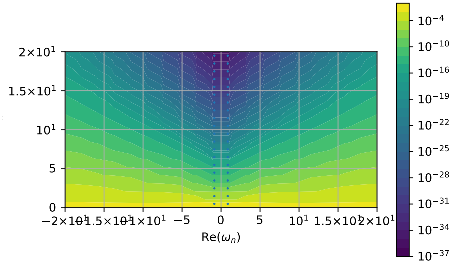
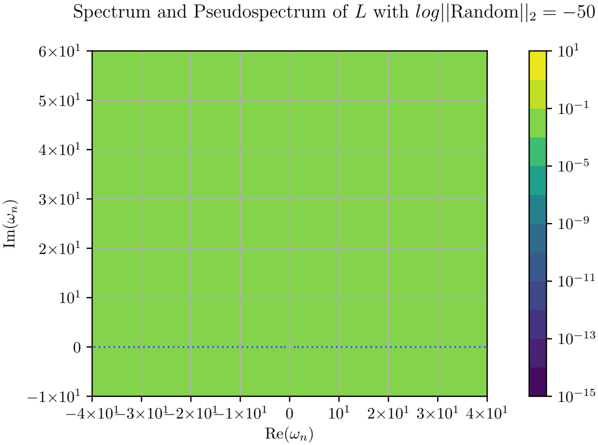
Poschel-teller Spectrum and Pseudospectrum of L



Test-bed study: Pöschl-Teller potential

Consistency check: self-adjoint case $\hat{L}_2 = 0$.



$$\tilde{V} = \tilde{V}_{\text{Poeschl}-\text{Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\boxed{\epsilon = 0}$$

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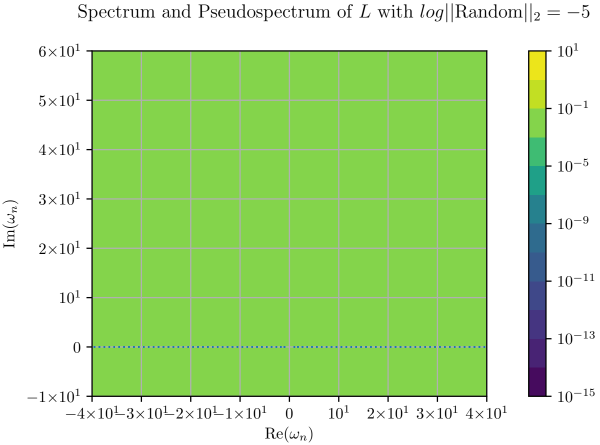

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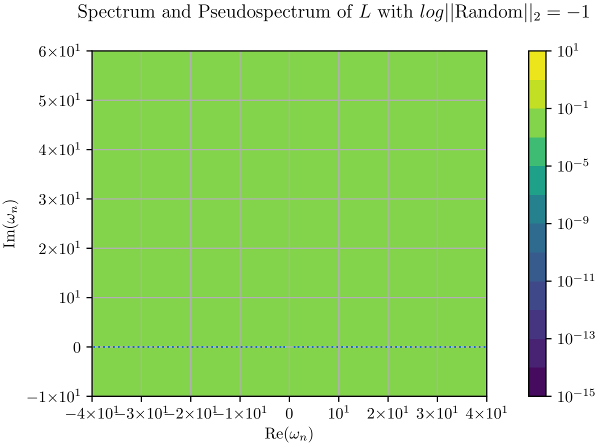
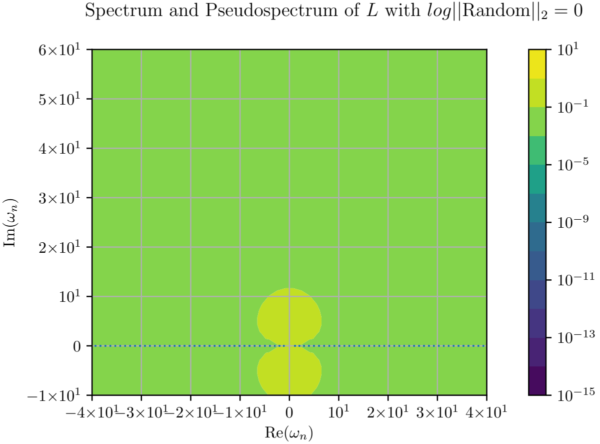


$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-1}}$$

Test-bed study: Pöschl-Teller potential

Consistency check: self-adjoint case $\hat{L}_2 = 0$.

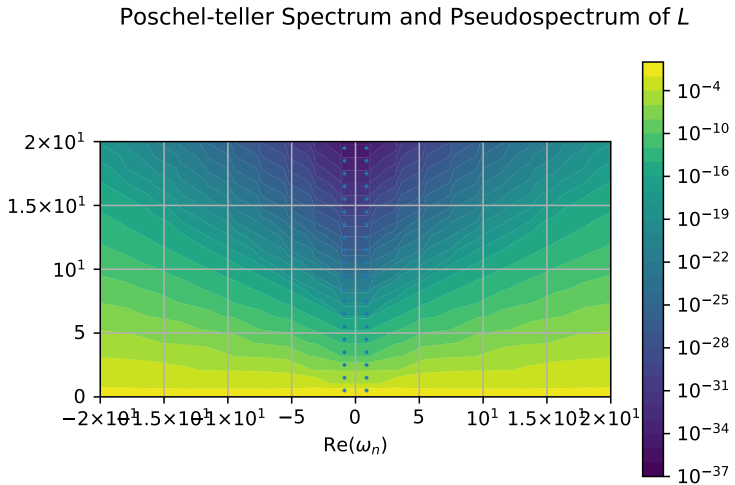


$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\boxed{\epsilon = 10^0}$$

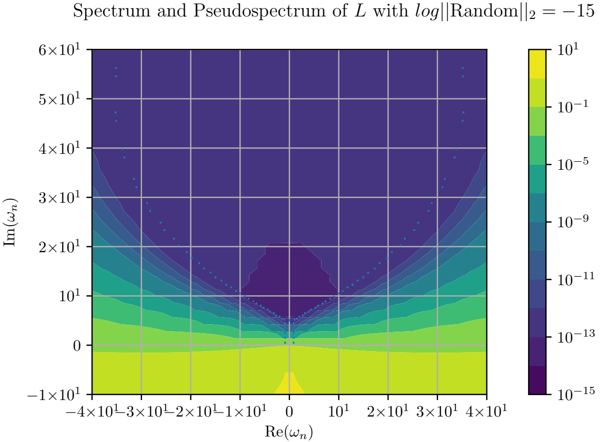
Test-bed study: Pöschl-Teller potential

QNM problem: $\hat{L}_2 \neq 0$.



Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

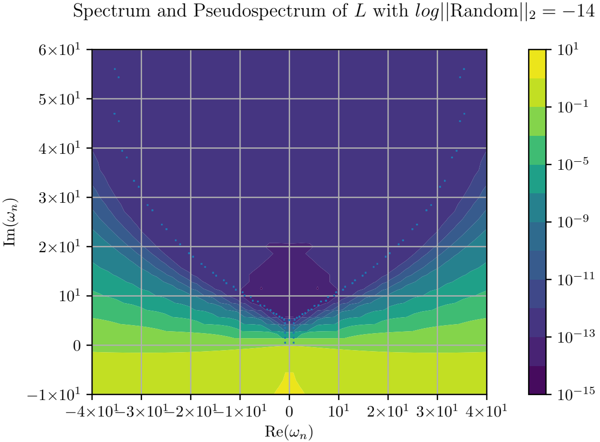


$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\epsilon = 10^{-15}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

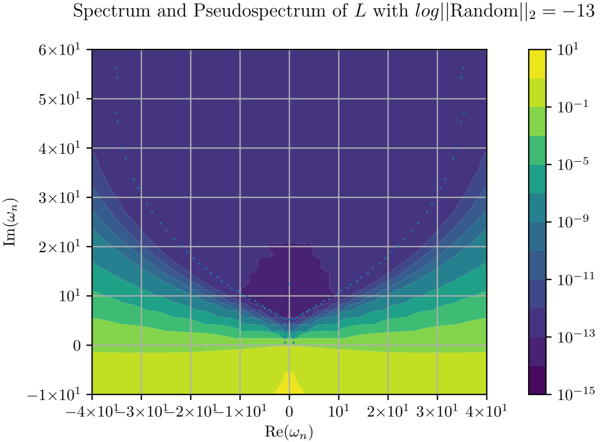


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\epsilon = 10^{-14}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.



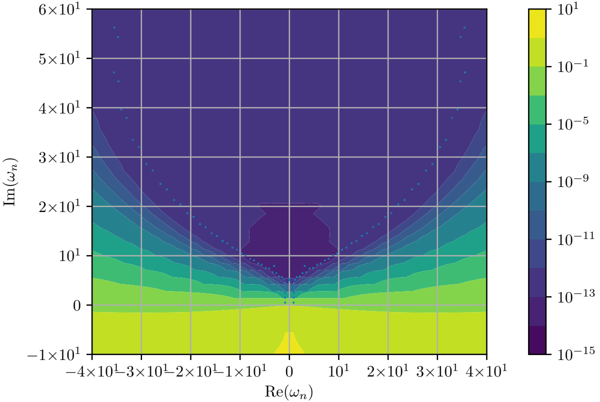
$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-13}}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

Spectrum and Pseudospectrum of L with $log||\mathrm{Random}||_2 = -12$

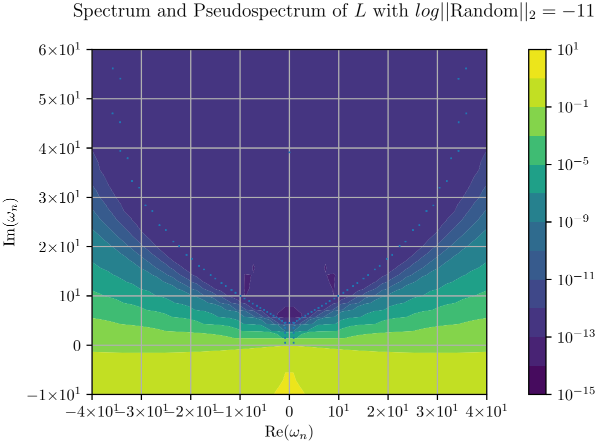


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\epsilon = 10^{-12}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

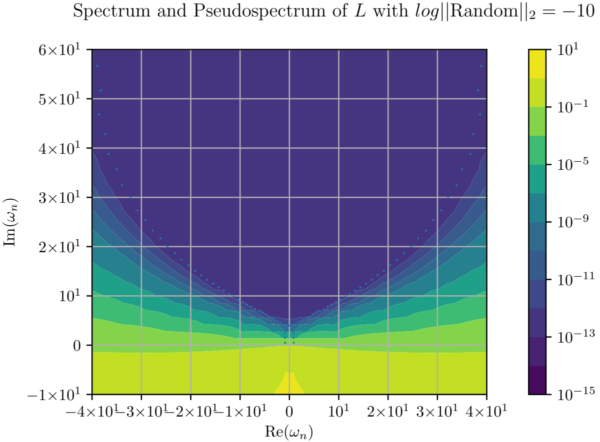


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\epsilon = 10^{-11}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.



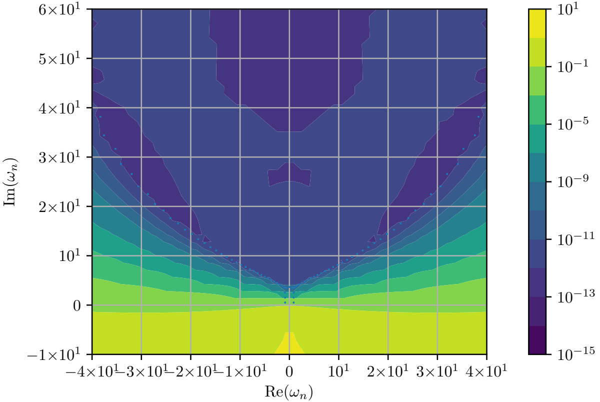
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-10}}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

Spectrum and Pseudospectrum of L with $log||\text{Random}||_2 = -9$



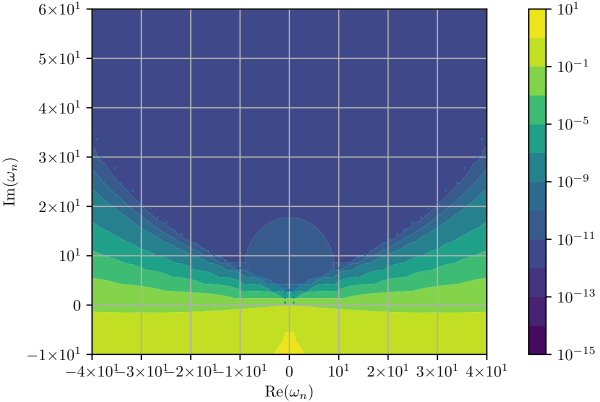
$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-9}}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

Spectrum and Pseudospectrum of L with $log||\text{Random}||_2 = -8$

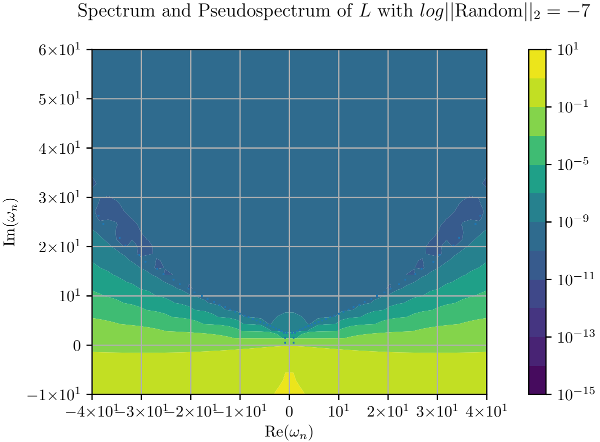


$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-8}}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

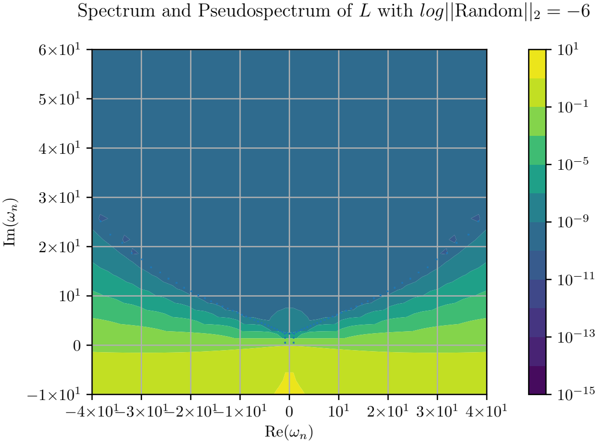


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\epsilon = 10^{-7}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

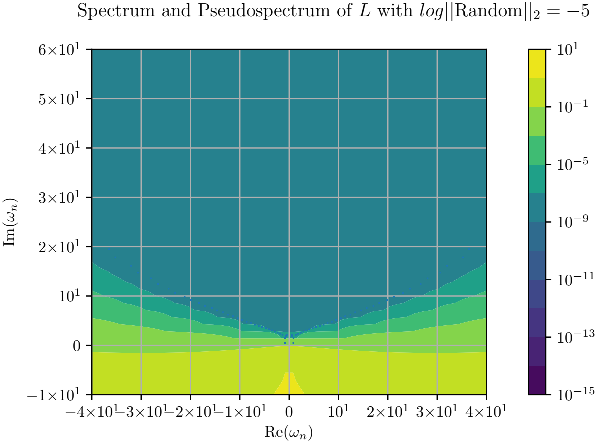


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-6}}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

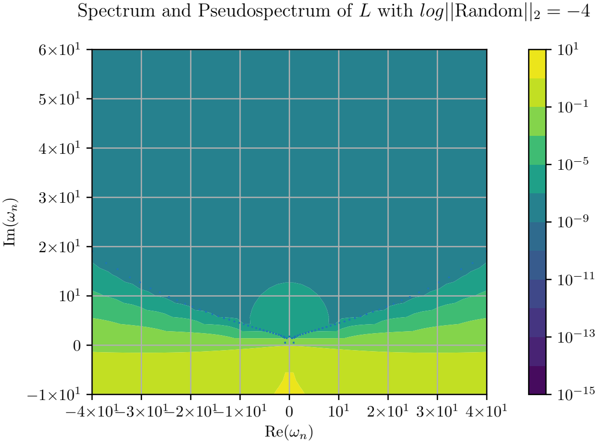


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\epsilon = 10^{-5}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

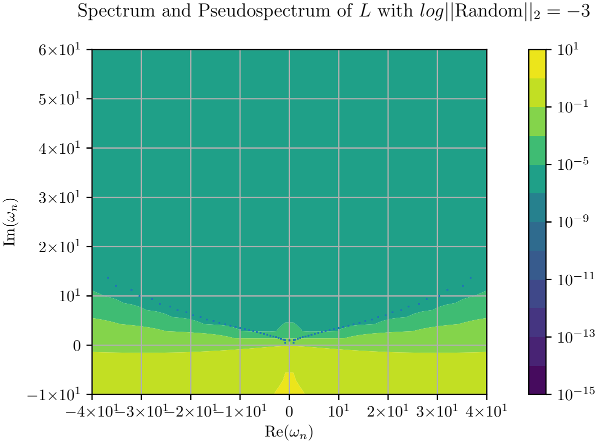


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\epsilon = 10^{-4}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

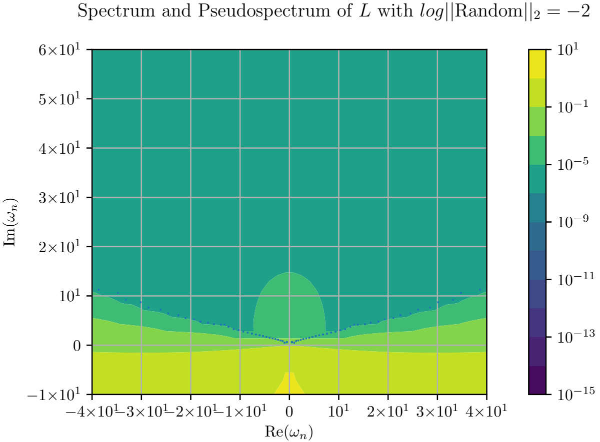


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-3}}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

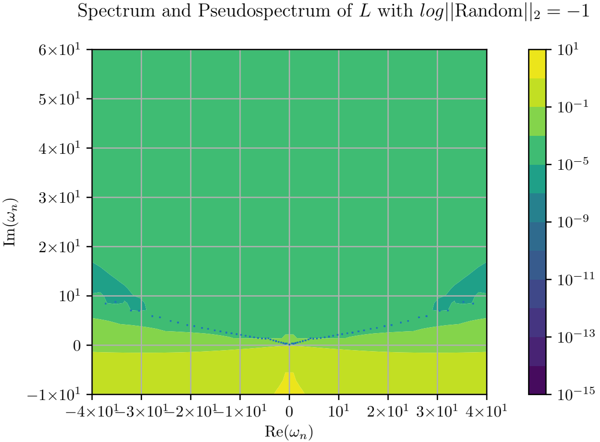


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-2}}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.

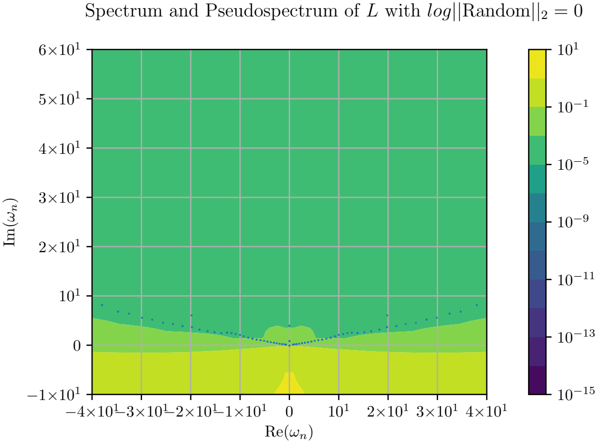


$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\boxed{\epsilon = 10^{-1}}$$

Test-bed study: Pöschl-Teller potential

QNM problem: random perturbation (in the potential). Fixed resolution.



$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$$\epsilon = 10^0$$

Test-bed study: Pöschl-Teller potential

From this we learn:

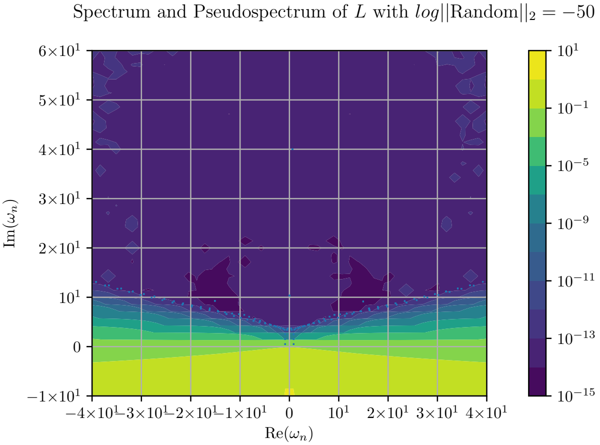
- **ALL overtones QNMs, unstable under high frequency perturbations**: instability grows as damping grows [JLJ, Macedo & Al Sheikh 21].
- Perturbations make **QNMs to "migrate" towards Pseudospectrum contour lines** ("extended pattern", cf. Bauer-Fike theorem).
- **Slowest damped QNM, stable under high frequency perturbations**:
  - Directly from the Pseudospectrum.
  - From the size of the needed perturbations.
- It can be repeated with deterministic **high frequency** k perturbations.
- For low frequency perturbations: much milder effect.

$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon \cos(2\pi k y) \quad , \quad k \gg 1$$

- It can also be seen: **Slowest damped QNM unstable under** "infrared perturbations" (cutting V at larges distances): **Nollert's instability of the fundamental QNM**.

Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution N. Fixed random perturbation $\epsilon = 10^{-16}$.

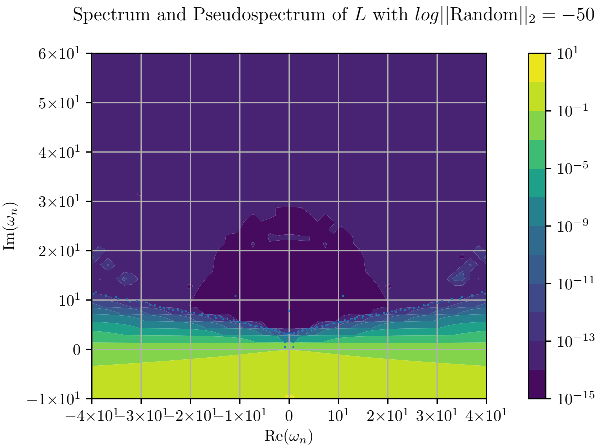


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$N = 120$$

Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution N. Fixed random perturbation $\epsilon = 10^{-16}$.

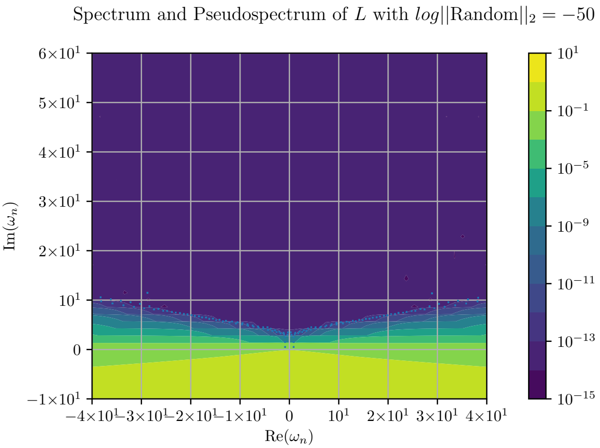


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$N = 140$

Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution N. Fixed random perturbation $\epsilon = 10^{-16}$.

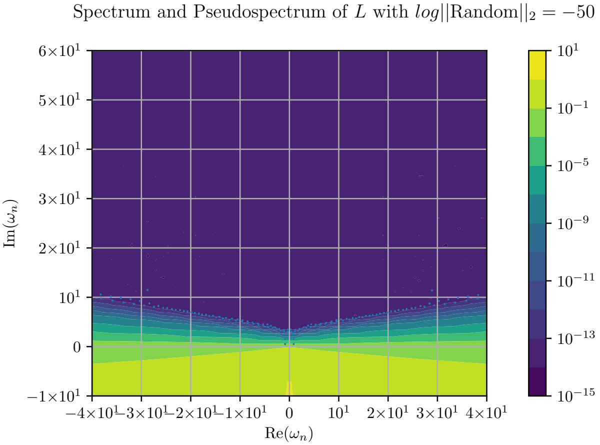


$$\tilde{V} = \tilde{V}_{\text{Poeschl-Teller}} + \epsilon E_{\text{Random}}, \quad ||E_{\text{Random}}|| = 1$$

$N = 160$

Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution N. Fixed random perturbation $\epsilon = 10^{-16}$.

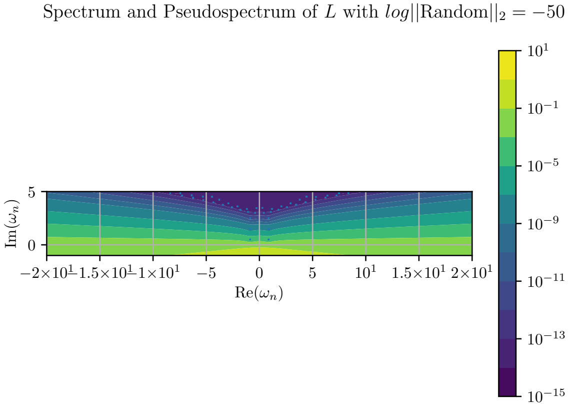


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$N = 160$

Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution N. Fixed random perturbation $\epsilon = 10^{-16}$.

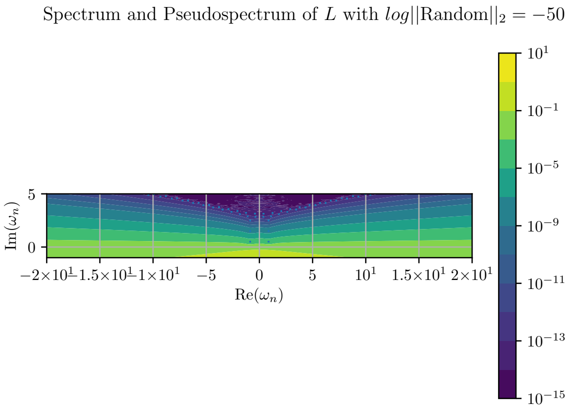


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\boxed{N = 160}$$

Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution N. Fixed random perturbation $\epsilon = 10^{-16}$.

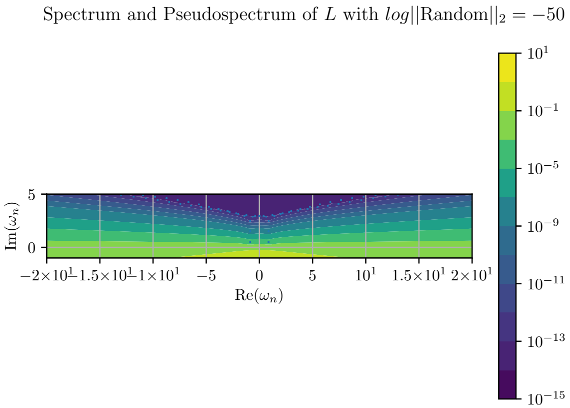


$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

$$\boxed{N = 180}$$

Test-bed study: Pöschl-Teller potential

QNM problem: Increasing resolution N. Fixed random perturbation $\epsilon = 10^{-16}$.



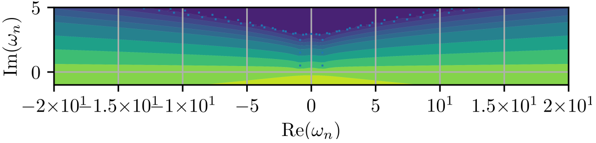
$$\tilde{V} = \tilde{V}_{\mathrm{Poeschl-Teller}} + \epsilon E_{\mathrm{Random}}, \quad ||E_{\mathrm{Random}}|| = 1$$

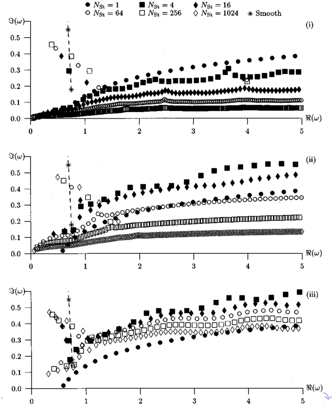
$$\boxed{N = 200}$$

Test-bed study: Pöschl-Teller potential

From this we learn:

- Pseudospectrum contour lines seem to converge to QNM curves observed by Nollert [cf. Nollert 96].
- Similarity to Neutron-Star "w-modes".





Test-bed study: Pöschl-Teller potential

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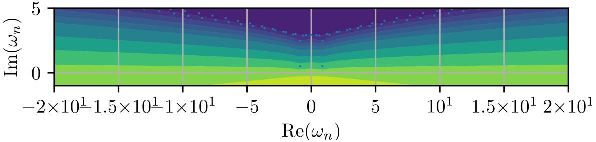
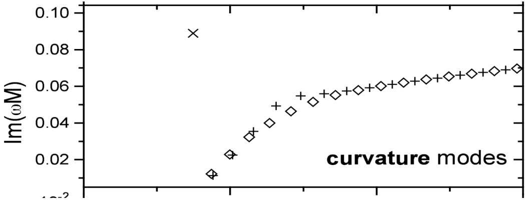
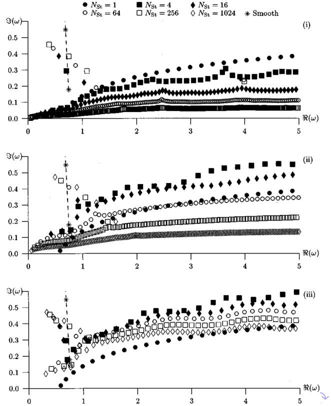


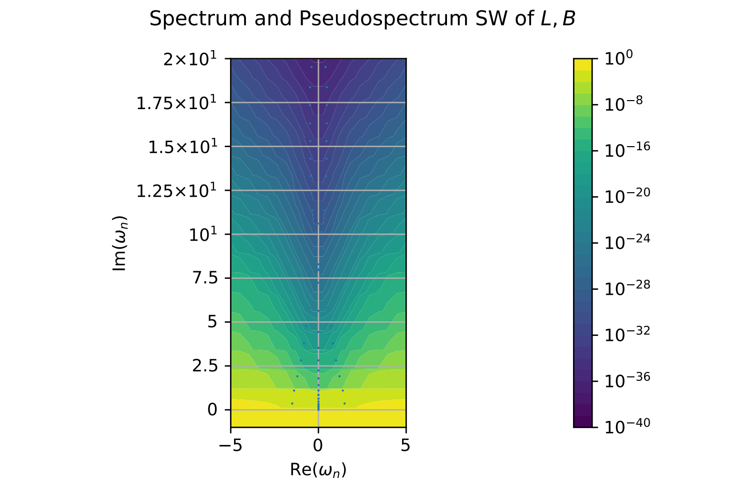


Curvature/w-modes in neutron stars [Kokkotas & Schmidt 99]



Schwarzschild QNMs

Schwarzschild Pseudospectum: same qualitative behaviour (note: branch cut)



Highly damped QNMs unstable, slowest decaying QNMs stable.

QNM: (spherically symmetric) general case [JLJ, Macedo, Al Sheikh 21]

Starting point: (scalar) wave equation in "tortoise" coordinates

On a stationary spatime (with timelike Killing ∂_t):

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V_\ell\right)\phi_{\ell m} = 0 \ ,$$

Dimensionless coordinates: $\bar{t} = t/\lambda$ and $\bar{x} = r_*/\lambda$ (and $\bar{V}_\ell = \lambda^2 V_\ell$),

Conformal hyporboloidal approach

$$\left\{ \begin{array}{lcl} \bar{t} & = & \tau - h(x) \\ \bar{x} & = & f(x) \end{array} \right. \quad .$$

- $h(x)$: implements the hyperboloidal slicing, i.e. $\tau = \mathrm{const.}$ is a horizon-penetrating hyperboloidal slice Σ_τ intersecting future $\mathscr{I}^+$.
- $f(x)$: spatial compactification between $\bar{x} \in [-\infty, \infty]$ to $[a, b]$.
- Timelike Killing: $\lambda\partial_t = \partial_{\bar{t}} = \partial_\tau$.

QNM: (spherically symmetric) general case [JLJ, Macedo, Al Sheikh 21]

First-order reduction: $\psi_{\ell m} = \partial_\tau \phi_{\ell m}$

$$\boxed{\partial_\tau u_{\ell m} = iLu_{\ell m}} \quad , \quad \text{with} \quad u_{\ell m} = \begin{pmatrix} \phi_{\ell m} \\ \psi_{\ell m} \end{pmatrix}$$

where

$$\boxed{L = \frac{1}{i}\left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array}\right)}$$

$$\begin{aligned} L_1 &= \frac{1}{w(x)}\left(\partial_x\left(p(x)\partial_x\right) - q(x)\right) \qquad \text{(Sturm-Liouville operator)} \\ L_2 &= \frac{1}{w(x)}\left(2\gamma(x)\partial_x + \partial_x\gamma(x)\right) \end{aligned}$$

with $w(x) = \dfrac{f'^2 - h'^2}{|f'|} > 0$, $p(x) = \dfrac{1}{|f'|}$, $q(x) = |f'|\, V_\ell$, $\gamma(x) = \dfrac{h'}{|f'|}$.

QNM: (spherically symmetric) general case [JLJ, Macedo, Al Sheikh 21]

Spectral problem

Taking Fourier transform, dropping (ℓ, m) (convention $u(\tau, x) \sim u(x)e^{i\omega\tau}$):

$$\boxed{L\, u_n = \omega_n\, u_n}\ .$$

where

$$\boxed{L = \frac{1}{i}\left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array}\right)}$$

$$\begin{aligned} L_1 &= \frac{1}{w(x)}\left(\partial_x\left(p(x)\partial_x\right) - q(x)\right) \qquad \text{(Sturm-Liouville operator)} \\ L_2 &= \frac{1}{w(x)}\left(2\gamma(x)\partial_x + \partial_x\gamma(x)\right) \end{aligned}$$

Conformal hyperboloidal approach: **No boundary conditions**

It holds $p(a) = p(b) = 0$, L_1 is "singular": **BCs "in-built" in** L.

QNM: (spherically symmetric) general case [JLJ, Macedo, Al Sheikh 21]

Scalar product

Natural scalar product (where $\tilde{V}_\ell := q(x) > 0$):

$$\langle u_1, u_2 \rangle_E = \frac{1}{2}\int_a^b \Big(w(x)\bar{\psi}_1\psi_2 + p(x)\partial_x\bar{\phi}_1\partial_x\phi_2 + \tilde{V}_\ell\bar{\phi}_1\phi_2 \Big) dx \ ,$$

associated with the "total energy" of ϕ on Σ_t, defining the **"energy norm"**

$$||u||^2_E = \langle u, u \rangle_E = \int_{\Sigma_\tau} T_{ab}(\phi, \partial_\tau\phi) t^a n^b d\Sigma_\tau \ ,$$

Spectral problem of a non-selfadjoint operator

- Full operator L: not selfadjoint.
- L_2: dissipative term encoding the energy leaking at $\mathscr{I}^+$.
- L selfadjoint in the non-dissipative $L_2 = 0$ case.

Non-normal operators spectral tools: "energy norm" for Pseudospectrum.

Application to Pöschl-Teller

Conformal compactification

$$\left\{ \begin{array}{rcl} \bar{t} & = & \tau - \frac{1}{2}\ln(1-x^2) \\ \bar{x} & = & \mathrm{arctanh}(x) \end{array} \right. \Leftrightarrow \left\{ \begin{array}{rcl} \tau & = & \bar{t} - \ln\left(\cosh\bar{x}\right) \\ x & = & \tanh\bar{x} \end{array} \right.$$

mapping $[-\infty, \infty]$ to $[a, b] = [-1, 1]$.

Spectral problem

Operators in L, with potential $V(x) = V_o \mathrm{sech}^2(\bar{x})$ (with $V_o = 1$):

$$\begin{array}{rcl} L_1 & = & \partial_x\left((1-x^2)\partial_x\right) - 1 \\ L_2 & = & -\left(2x\partial_x + 1\right) \ . \end{array}$$

where:

$$w(x) = 1 \ \ , \ \ p(x) = (1-x^2) \ \ , \ \ q(x) = \frac{V}{1-x^2} =: \tilde{V}(x) \ \ , \ \ \gamma(x) = -x.$$

Application to Schwarzschild

Schwarzschild potential

Axial (Regge-Wheeler) case (also for polar (Zerilli) parity):

$$V_\ell^s = \left(1 - \frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} + (1-s^2)\frac{2M}{r^3}\right) ,$$

where

- $r_* = r + 2M\ln(r/2M - 1)$
- $s = 0, 1, 2$, respectively, to the scalar, electromagnetic and gravitational cases.

Numerical Chebyshev methods: analyticity of $V(x)$

Bizoń-Mach coordinates used in Pöschl-Teller not well adapted now: the potential is non-analytic in x, spoiling the accuracy of Chebyshev's methods.
Same problem for the polar (Zerilli) case.

Solution

We resort rather to the 'minimal gauge' hyperboloidal slicing [Ansorg, Macedo 16; Macedo 18] guaranteeing the analyticity of the Schwarzschild potential.

Application to Schwarzschild

Conformal compactification: "minimal gauge" [Ansorg, Macedo 16; Macedo 18]

$$\left\{ \begin{array}{rcl} \bar{t} & = & \tau - \frac{1}{2}\left(\ln\sigma + \ln(1-\sigma) - \frac{1}{\sigma}\right) \\ \bar{x} & = & \frac{1}{2}\left(\frac{1}{\sigma} + \ln(1-\sigma) - \ln\sigma\right) \end{array} \right. ,$$

mapping $[-\infty, \infty]$ to $[a,b] = [0,1]$ (we use σ, rather than x).

Spectral problem

Operators in L:

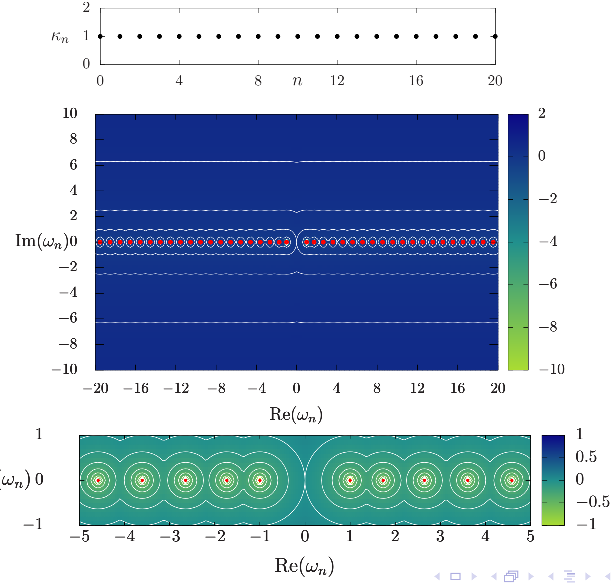
$$\begin{array}{rcl} L_1 & = & \dfrac{1}{1+\sigma}\left[\partial_\sigma\left(\sigma^2(1-\sigma)\partial_\sigma\right) - \left(\ell(\ell+1) + (1-s^2)\sigma\right)\right] \\ L_2 & = & \dfrac{1}{1+\sigma}\left((1-2\sigma^2)\partial_\sigma - 2\sigma\right) \ . \end{array}$$

where:

$$w(\sigma) = 1+\sigma \ , \ p(\sigma) = \sigma^2(1-\sigma) \ , \ q(\sigma) = \frac{V}{\sigma^2(1-\sigma)} =: \tilde{V}(\sigma) \ , \ \gamma(\sigma) = 1 - 2\sigma^2.$$

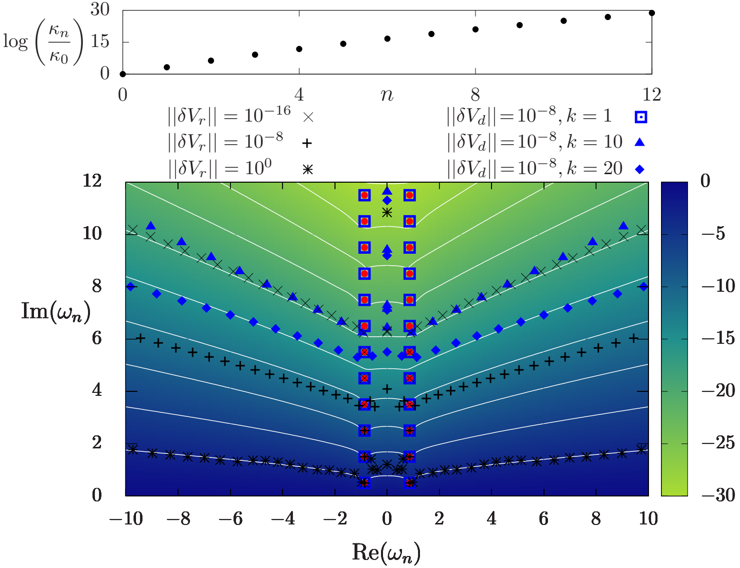
Trying to understand: the "current picture"... in pictures

Pöschl-Teller, $L_2 = 0$:



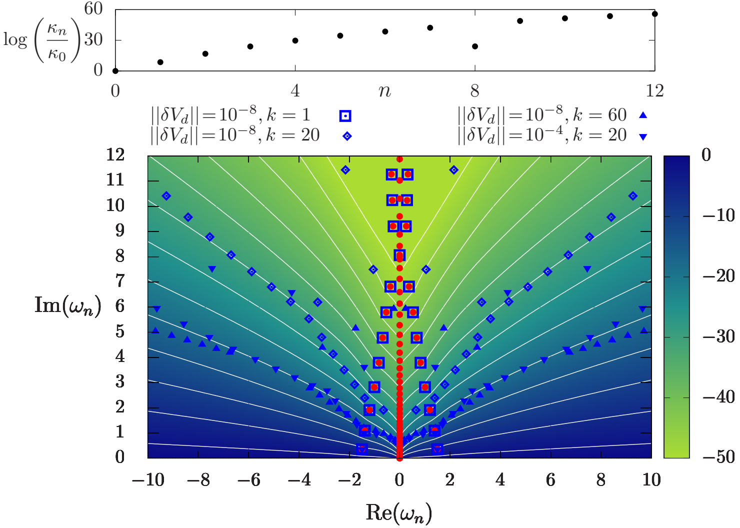
Trying to understand: the "current picture"… in pictures

Pöschl-Teller, $L_2 \neq 0$:



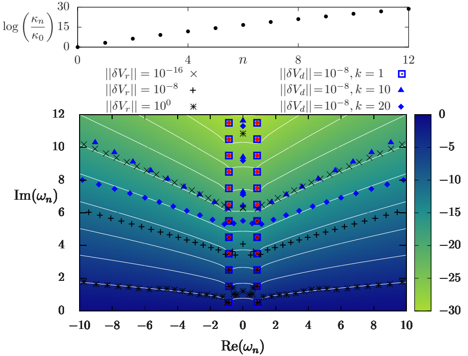
Trying to understand: the "current picture"... in pictures

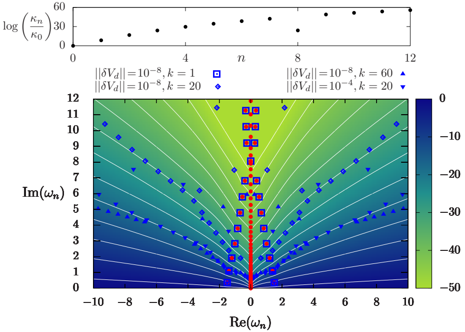
Schwarzschild ($s = 2$, $\ell = 2$):



Trying to understand: the "current picture"... in pictures

Comparison Pöschl-Teller versus Schwarzschild:

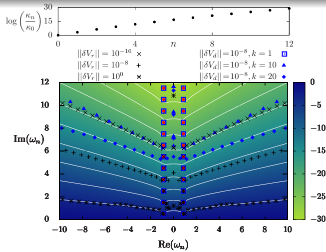


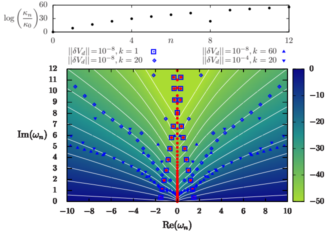


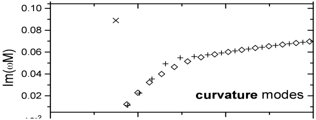
Remarks:

- High frequency perturbations: random δV_r, deterministic $\delta V_d \sim \cos(2\pi k x)$.
- **Fundamental QNM stable**.
- **Perturbed QNMs "migrate" towards $\epsilon-$contour lines of Pseudospectra.**
- **'Universality' phenomenon?**

Trying to understand: the "current picture"... in pictures



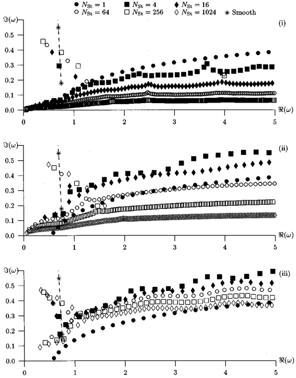




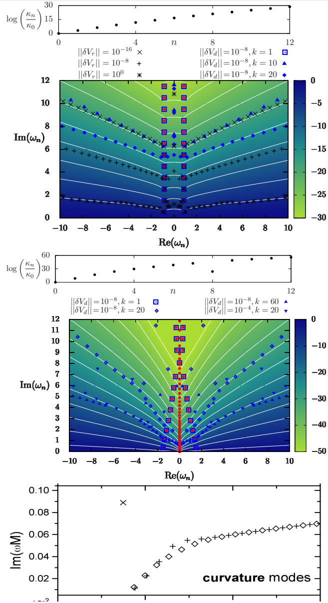
Black Hole and Neutron Star QNMs

Comparison with:

- Nollert's high-frequency Schwarzschild perturbations.
- Nollert's remark on Neutron Stars (w-modes) curvature QNMs.



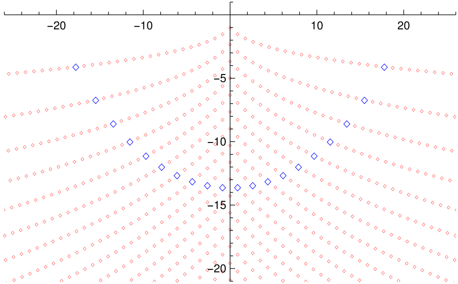
Trying to understand: the "current picture"... in pictures



"Duality" QNMs (long-range potentials) and Regge poles (compact support potentials)?

QNM of an spherical obstacle [Stefanov 06]:

- Red-diamonds: fixed "n", running angular ℓ.
- Blue-diamonds: fixed ℓ (here $\ell = 20$), running n.



Scheme

Connecting with Lax-Phillips: Keldysh expansion

Hilbert space: Keldysh's expansion of the resolvent [Keldysh 51, 71]

Given v_j and w_j (right- . left and right eigenvectors of L), with $\langle w_n, v_n\rangle = -1$, the resolvent of L can be expanded in a domain Ω around the poles as

$$(L-\lambda)^{-1} = \sum_{\lambda_j \in \Omega} \frac{|v_j\rangle\langle w_j|}{\lambda - \lambda_j} + H(\lambda) \quad , \quad \lambda \in \Omega \setminus \sigma(L)$$

QNM resonant expansions, "1st-order time reduction" [Al Sheikh, JLJ, Gasperin 21, 22]

The field u satisfying $\partial_\tau u = iLu$, with $u(\tau = 0, x) = u_0$, can be written in an "asymptotic expansion" as

$$u(\tau, x) \;=\; \sum_{j=1}^{N} e^{i\omega_j \tau} \kappa_j \langle \hat{w}_j | u_0 \rangle_{_E} \hat{v}_j + E_N(\tau; S)$$

with $\quad ||E_N(\tau;S)||_{_E} \leq C_N(a, L) e^{-a\tau} ||S||_{_E}$

Note the presence of the condition number κ_j.

Connecting with Lax-Phillips: Keldysh expansion

Hilbert space: Keldysh's expansion of the resolvent [Keldysh 51, 71]

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QNM resonant expansions, "scattered field" [Al Sheikh, JLJ, Gasperin 21, 22]

In terms of the scattered field satisfying the 2nd order equation

$$\phi(\tau, x) \sim \sum_n e^{i\omega_j \tau} a_n \hat{\phi}^R_j(x) \quad , \quad \phi(t, x) \sim \sum_n e^{i\omega_j t} a_n e^{i\omega_j h(x)} \hat{\phi}^R_j(x)$$

$$\text{with} \quad a_j = \frac{\kappa_j}{2}\left(\langle \hat{\phi}^L_j, \varphi_0 \rangle_{H^1_{(V,p)}} - i\omega_j \langle \hat{\phi}^L_j, \varphi_1 \rangle_{(2,w)} \right)$$

that recovers the expression in terms of normal modes in the selfadjoint (normal) case: $\kappa_j = 1$, $\hat{\phi}^R_j(x) = \hat{\phi}^L_j(x)$.

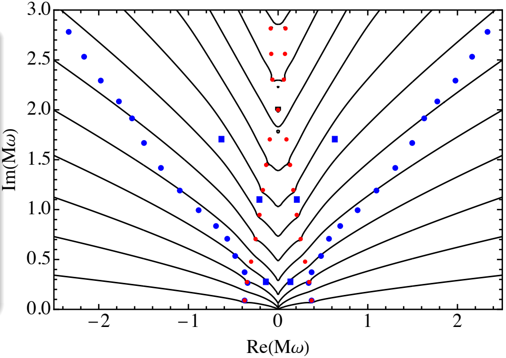
Are the perturbed QNMs in the ringdown signal? Yes

Extracting QNM: Mode Filtering

[JLJ, Macedo, Al Sheikh 21]

$$\Phi^N_{\rm spec}(t) := \sum_{n=0}^{N} \mathcal{A}_n e^{i\omega_n t}$$

$$\mathcal{F}^N(t) = \Phi_{\rm evol}(t) - \Phi^N_{\rm spec}(t)$$



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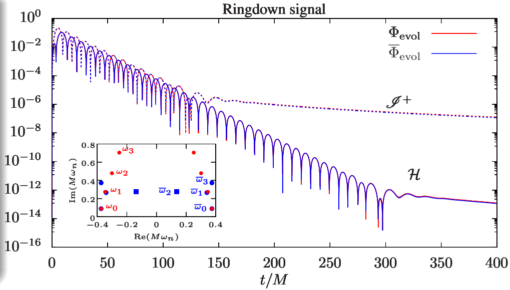

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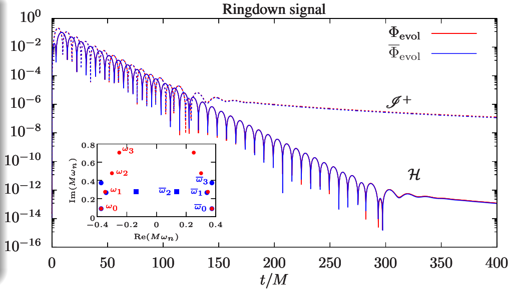
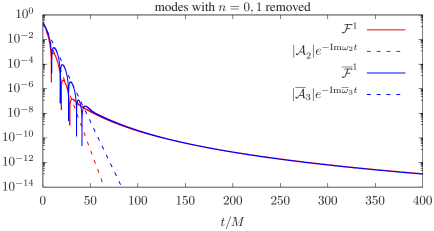
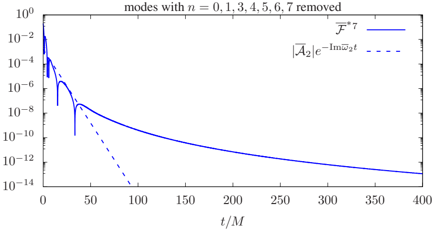


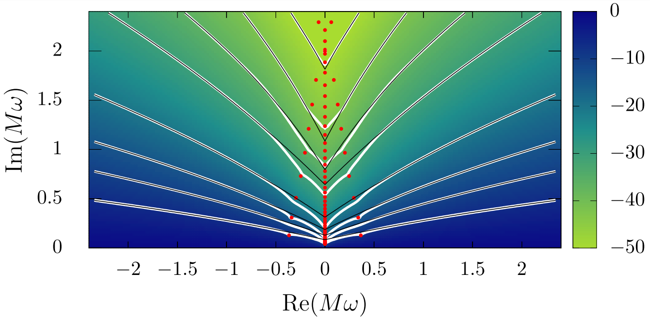




Emerging picture: "Universality" of compact-object QNMs

Logarithm QNM-free regions: Pseudospectrum contour lines [JLJ, Macedo, Al Sheikh 21]

- QNM overtones: ultraviolet (high-frequency) unstable.
- Fundamental QNM: ultraviolet (high-frequency) stable.
  (Warning: "Flea on the Elephant" effect for "trapping potentials" [Jona-Lasinio et al. 81, 84; Helffer & Sjöstrand 83, 85; Simon 85; Cheung et al. 21,...])
- Logarithmic $|\mathrm{Re}(\omega)| \gg 1$ asymptotics: $\mathrm{Im}(\omega) \sim C_1 + C_2 \ln(|\mathrm{Re}(\omega)| + C_3)$



Emerging picture: "Universality" of compact-object QNMs

C^p regularity: Regge QNM branches, logarithmic asymptotics

- Compact support potential: **theorem** by Zworski (Weyl law) [Zworski 87] (Note: this **fully** explains Nollert's results [Nollert 96]
- Non-compact support, $C^{p<\infty}$ potential: WKB Berry's treatment [Berry 82].
- Related results by Nollert-Price [Nollert & Price 99]
- Polytropic neutron stars: w-modes [Zhang, Wu & Leung 11]

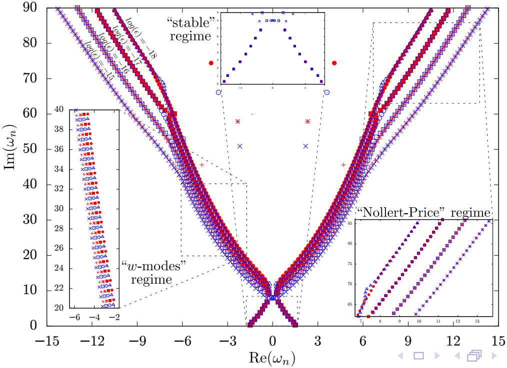
$$
\begin{aligned}
\mathrm{Re}(\omega_n) &\sim \pm\left(\frac{\pi}{L}n + \frac{\pi\gamma_p}{2L}\right) \\
\mathrm{Im}(\omega_n) &\sim \frac{1}{L}\left[\gamma \ln\left(\left(\frac{\pi}{L}n + \frac{\pi\gamma_p}{2L}\right) + \frac{\pi\gamma_\Delta}{2L}\right) - \ln S\right]
\end{aligned}
$$

- *"Length scale"* L *"Small structure" coefficient* γ, *"Strength coefficient"* S.
- *"Real shift"* γ_Δ.
- *"Parity term"* γ_p.

Emerging picture: "Universality" of compact-object QNMs

Regular perturbations: "Nollert-Price-like" branches and "inner QNMs"

- Always above the logarithm pseudospectra lines: "Lamis' Graal".
- "Nollert-Price-like": different branches, transitions at critical values.
- "Inner modes": appearing at the critical values. Weyl law.
- Nollert-Price branches towards logarithm pseudospectra as perturbation frequency increases: **Do they get to the logarithmic pseudospectra lines?**



From Burnett conjecture to a Regge QNM conjecture

Burnett's conjecture: high-frequency limit of spacetimes [Burnett 89; Huneau, Luk 18, 19]

The limit of high-frequency oscillations of a vacuum spacetime is effectively described by an effective matter spacetime described by massless Einstein-Vlasov.

Luk & Rodnianski: "null shells" as low-regularity problems in Einstein equations

- Spikes are more efficient in triggering instabilities [Gasperin & JLJ 21]:

$$|\delta g_{ab}| \, ||\delta L||_{E} \sim \max_{\mathrm{supp}(\delta g)} |\partial \delta g_{ab}|^2 \sim \max_{\mathrm{supp}(\delta g)} \rho_E(\delta g; x)$$

- Luk & Rodnianski's treatment [Luk & Rodnianski 20]: Allowing for "concentrations", the infinite high-frequency limit is a **low regularity limit**

$$g_n \to g_\infty \ \text{ in } C^0 \cap H^1 \ , \ \partial g_n \to \partial g_\infty \ \text{ in } L^2$$

Regge QNM branches conjecture: a low-regularity problem [JLJ, Macedo, AL Sheikh 21, Gasperin & JLJ 21]

In the limit of infinite frequency, generic ultraviolet perturbations push BH QNMs in Nollert-Price branches to asymptotically logarithmic branches along the QNM-free region boundary, exactly following the Regge QNM asymptotic pattern.

QNMs and norms: "Definition" versus "Stability" problems

Definition of QNMs: we can (need to) choose the norm to control QNMs

- Ansorg & Macedo [Ansorg & Macedo 16]: if C^∞-regularity, then every point in the upper complex plane is an eigenvalue. Need of "more control".
- Warnick & Gajic [Warnick & Gajic 19]: Fundamental contribution by identifying in **Gevrey-2** regularity classes in (**asymptotically flat**) hyperboloidal framework (complemented in [Galkowski & Zworski 20] in a complex scaling setting).
  Hilbert spaces of $(\sigma, 2)$-Gevrey functions on $[0, \rho_0]$: f, $g \in C^\infty((0, \rho_0); \mathbb{C})$

$$\langle f, g \rangle_{G^2_{\sigma,1,\rho_o}} = \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{n!^2 (n+1)!^2} \int_0^{\rho_0} \partial^n_\rho f \partial^n_\rho \bar{g} d\rho$$

Stability of QNMs: we cannot choose the norm, the "physics" chooses for us

The system chooses what is a "big and small" perturbation: **here, the energy**.

$$||(\phi, \psi)||_E = E = \int_\Sigma T_{ab} t^a n^b d\Sigma \sim ||\phi||_{H^1_V} + ||\psi||_{L^2}$$

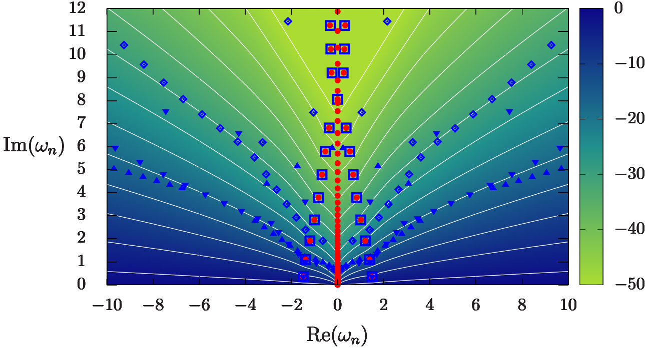
"Energy-Pseudospectrum = Chart of the Gevrey Ocean"

QNMs and norms: “Definition” versus “Stability” problems

Towards a reconciliation: the Gevrey Ocean

A small high-frequency perturbation in the energy, is a huge one in Gevrey terms.

- For a given V, **QNM eigenfunctions are defined to be in class Gevrey-2**.
- **Small (in the energy norm) perturbations** δV **of** V **lead to very different QNM** ω_n**'s**, respective eigenfunctions being indeed Gevrey-2.



“Energy-Pseudospectrum = Chart of the Gevrey Ocean”

Conclusions and Perspectives

Conclusions

- Numerical evidence of instability of QNM overtones under high-frequency perturbations in the effective potential.
- Independent support from Pseudospectrum of non-perturbed operator.
- Hints towards **Universality of compact-object QNM branches** in the infinite high-frequency limit: a low regularity problem.

Perspectives

- **Can we measure the regularity of spacetime?**
  [paraphrasing: *"What is the regularity of spacetime?"* [question from Bruce Allen]]
- To transform presented heuristic and numerical evidences into Theorems:
  i) Semiclassical estimates of the pseudospectrum.
  ii) Regge QNM conjecture from Burnett's conjecture.
- Implement Pseudospectrum in Gevrey-2 scalar products: trivialization?
- Study stability of the time domain signal under high-frequency perturbations.
- ...

Conclusions and Perspectives

The “Gevrey Ocean” and the Dijon Legacy...

Conclusions and Perspectives

"Truth suffers from much Analysis"

(Ancient Fremen saying) Dune Messiah, Frank Herbert

A 'hint of Geometry' perhaps...?