Seismic normal modes, Rayleigh waves, resonances and inverse problems

– reconciliation of seismology with analysis

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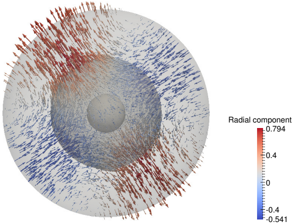
Reims 2021

terrestrial planets, discrete spectrum

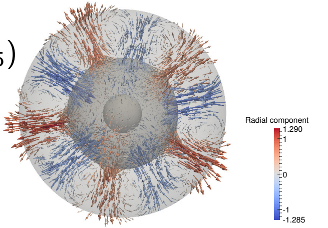
omitting rotation

seismic normal modes

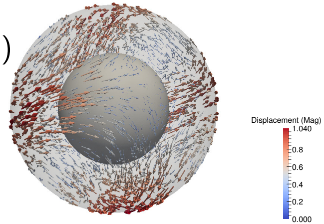
$({}_0S_2)$



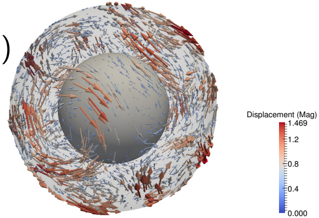
$({}_0S_5)$



$({}_0T_2)$



$({}_0T_5)$



decomposition of natural Hilbert space

radial manifold with boundary, $M = \overline{B(0,1)}$ – Riemannian metric

$$g(x) = c^{-2}(|x|)e(x), \quad c\colon (0,1] \to (0,\infty)$$

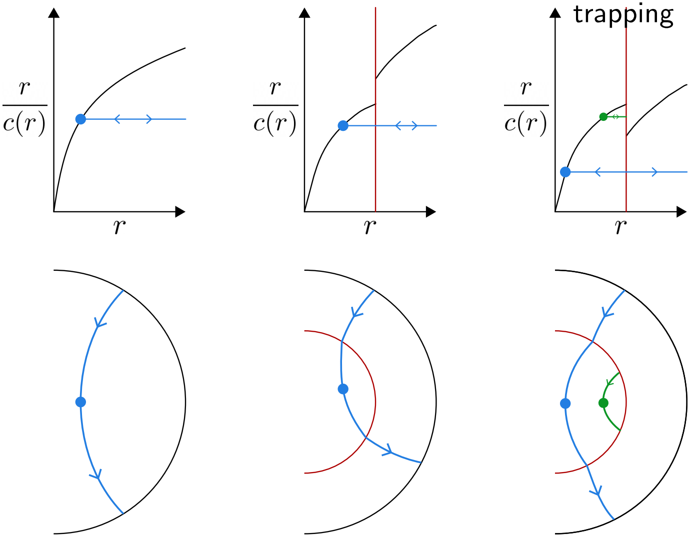
e is the standard Euclidean metric

$c(r)$ has a jump discontinuity at a finite set of values $r = r_1, \cdots, r_K$; that is $\lim_{r\to r_k^-} c(r) \neq \lim_{r\to r_k^+} c(r)$ for each i (annuli $A(r_{k-1}, r_k)$)

a *maximal geodesic* is a unit speed geodesic on the Riemannian manifold with each endpoint at its boundary or at an interface

a broken ray is a concatenation of maximal geodesics satisfying the reflection condition of geometrical optics at both inner and outer boundaries of M, and Snell's law for geometric optics at the interfaces

conditions



Herglotz condition

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r}{c(r)}\right) > 0$$

away from discontinuities

a broken ray is called *basic* if

- it stays within a single layer and all of its legs are reflections from a single interface, or
- it is a *radial* ray contained in a single layer; such a ray is defined to be a ray with zero epicentral distance and will necessarily reflect from two interfaces

let γ be a basic ray with radius R^* ($r_k \leq R^* < r_{k]1}$), (conserved) ray parameter p, which lies inside inside $A(r_{k-1}, r_k)$ $(1 = r_0 > r_1 > \cdots > r_K)$; there is a unique $N \in \mathbb{N}$ so that its length T is

$$T = 2NL_\gamma := 2N \int_{R^*}^{r_{k-1}} \frac{1}{c(r')^2 \beta(r'; p)} \, \mathrm{d}r', \quad \beta(r; p)^2 = c(r)^{-2} - r^{-2} p^2$$

and angular or epicentral distance

$$\alpha_\gamma := \alpha(p) = 2N \int_{R^*}^{r_{k-1}} \frac{p}{(r')^2 \beta(r'; p)} \, \mathrm{d}r'$$

conditions

Definition

Consider geodesics in an annulus $A(a, b)$ equipped with a $C^{1,1}$ wave speed $c\colon (a, b] \to (0, \infty)$. It satisfies the *countable conjugacy condition* if there are only countably many radii $r \in (a, b)$ so that the endpoints of the corresponding maximal geodesic $\gamma(r)$ are conjugate along that geodesic.

Definition

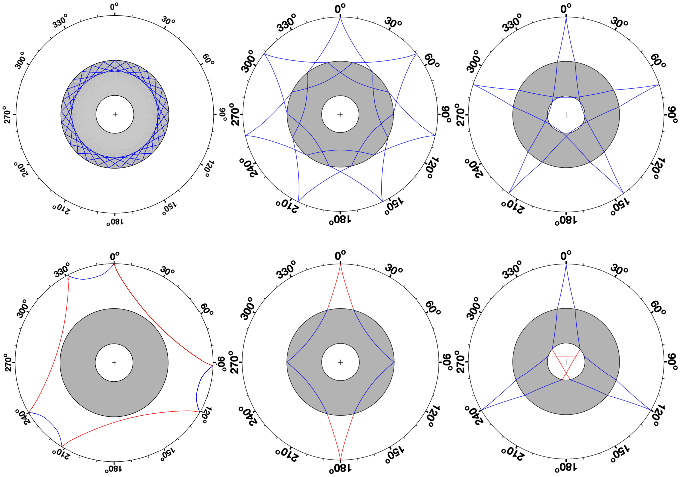
The radial wave speed c satisfies the *periodic conjugacy condition* if for each periodic, nongliding ray with a ray parameter p, $\partial_p \alpha(p) \neq 0$. (This ensures that the phase function in the stationary phase argument for computing the trace formula is Bott-Morse nondegenerate.)

$c_\tau :\ [0, 1] \to (0, \infty)$ indexed by $\tau \in (-\varepsilon, \varepsilon)$ is an "admissible" family of profiles

(basic) length spectrum

length spectrum, $\mathrm{lsp}(c)$: the set of lengths of all periodic broken rays

basic length spectrum : $\mathrm{blsp}(c)$



P in blue, *S* in red (PKPab, PKIKP, SP, SKKS, PKJKP)

conditions

- equivalence classes $[\gamma]$ (rotations, time reversal, dynamic analogs) parameterized by p
- $Q_{[\gamma]}$ is product of reflection and transmission coefficients (transmission conditions)
- $n_{[\gamma]}$ is number of dynamic analogs

Definition

The length spectrum satisfies the *principal amplitude injectivity condition* if given two closed rays γ_1 and γ_2 with the same period and disjoint equivalence classes (so they must have different ray parameters p_1 and p_2, then

$$n_{[\gamma_1]} Q_{[\gamma_1]} |p_1^{-2} \partial_p \alpha(p_1)|^{-1/2} \neq n_{[\gamma_2]} Q_{[\gamma_2]} |p_2^{-2} \partial_p \alpha(p_2)|^{-1/2}$$

ensuring recovery of T.

Theorem

Fix any $\varepsilon > 0$ *and* $K \in \mathbb{N}$, *and let* $c_\tau(r)$ *be an admissible family of profiles with discontinuities at* $r_k(\tau)$ *for all* $k = 1, \ldots, K$. *Let* $\mathrm{blsp}(\tau)$ *denote the basic length spectrum with the wave speed profile* c_τ. *Suppose* $\mathrm{blsp}(\tau)$ *is countable for all* τ. *Let* $S(\tau)$ *be any collection of countable subsets of* $\mathbb{R}$ *indexed by* τ.

If $\mathrm{blsp}(\tau) \cup S(\tau) = \mathrm{blsp}(0) \cup S(0)$ *for all* $\tau \in (-\varepsilon, \varepsilon)$, *then* $c_\tau = c_0$ *and* $r_k(\tau) = r_k(0)$ *for all* $\tau \in (-\varepsilon, \varepsilon)$ *and* $k = 1, \ldots, K$.

Corollary (Length spectral rigidity with two polarizations)

Fix any $\varepsilon > 0$ and $K \in \mathbb{N}$, and let $c^i_\tau(r)$ with both $i = 1, 2$ be an admissible family of profiles with discontinuities at $r_k(\tau)$ for all $k = 1, \ldots, K$. Consider all periodic rays which are geodesics within each layer and satisfy the usual reflection or transmission conditions at interfaces, but which can change between the wave speed profiles c^1_τ and c^2_τ at any reflection and transmission. Suppose that the length spectrum of this whole family of geodesics, denoted by $\operatorname{lsp}(\tau)$, is countable in the ball $\overline{B(0,1)}$.

If $\operatorname{lsp}(\tau) = \operatorname{lsp}(0)$ for all $\tau \in (-\varepsilon, \varepsilon)$, then $c^i_\tau = c^i_0$ for both $i = 1, 2$ and $r_k(\tau) = r_k(0)$ for all $\tau \in (-\varepsilon, \varepsilon)$ and $k = 1, \ldots, K$.

spectral rigidity, terrestrial planets

Theorem (Spectral rigidity with moving interfaces)

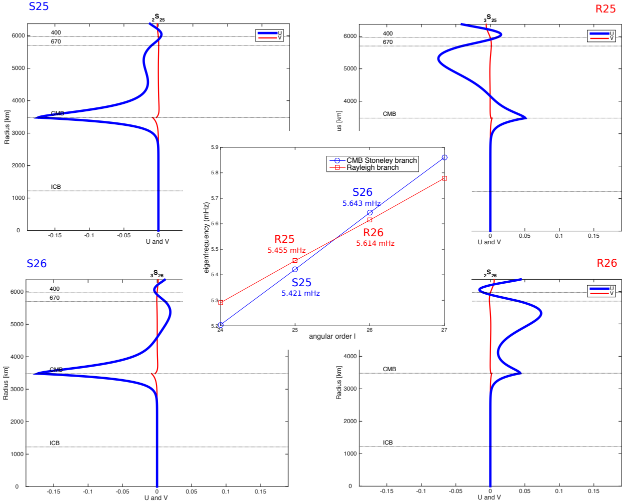
Fix any $\varepsilon > 0$ and $K \in \mathbb{N}$, and let $c_\tau(r)$ be an admissible family of profiles with discontinuities at $r_k(\tau)$ for all $k = 1, \ldots, K$. Suppose that the length spectrum for each c_τ is countable in the ball $\bar{B}(0,1) \subset \mathbb{R}^3$. Assume also that the length spectrum satisfies the principal amplitude injectivity condition and the periodic conjugacy condition.

Suppose $\operatorname{spec}(\tau) = \operatorname{spec}(0)$ for all $\tau \in (-\varepsilon, \varepsilon)$. Then $c_\tau = c_0$ and $r_k(\tau) = r_k(0)$ for all $\tau \in (-\varepsilon, \varepsilon)$ and $k = 1, \ldots, K$.

trace formula – possible periodic broken rays γ_0, say, with gliding

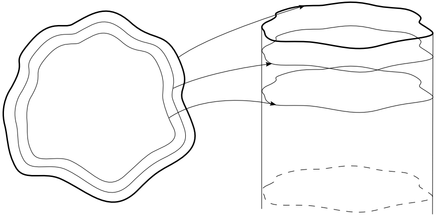
- gliding occurs at only one interface; this is ensured by the Herglotz condition
- there is a sequence of periodic non-gliding broken rays γ_i so that $\gamma_i \to \gamma_0$; subtlety lies in ensuring periodicity of the approximating rays

“near” phase boundaries, Earth’s surface
Love and Rayleigh waves: local recovery



Stoneley ⟶ ⟵ Rayleigh

Earth as a unit ball $B_1 = B(0,1)$; there is a global diffeomophism, ϕ



$$\phi : \ B_1 \setminus \{0\} \to S^2 \times \mathbb{R}^-$$

$$\phi(B_r) = S^2 \times \left\{1 - \frac{1}{r}\right\}, \ r \neq 0$$

- for an open and bounded subset $U \subset S^2$, the cone region, $\{(\Theta, r) \mid \Theta \in U, \ 0 < r < 1\}$, is diffeomorphic to $U \times \mathbb{R}^-$; we can find global coordinates for U and we may consider our system on the domain $S^2 \times \mathbb{R}^-$
- more generally, we consider the system on any Riemannian manifold of the form $M = \partial M \times \mathbb{R}^-$ with metric

$$g = \begin{pmatrix} g' & 0 \\ 0 & 1 \end{pmatrix}$$

- for a "nice" domain Ω, a neighborhood of the boundary is diffeomorphic to M, where the metric g' is the induced metric of the boundary of Ω

seismology – elastic wave equation

- Rayleigh waves/modes have long/widely been used to study Earth's crust and upper mantle (Dorman & Ewing, 1962)
- empirically it has been established that "phase velocities" or eigenvalues (fundamental mode and overtones) at a few discrete frequencies are insufficient data to determine both P- and S-wave speeds (Lamé parameters)
- it is now common practice to add data: "H/V" related to the components of the trace of modes, and information from body waves/modes

analysis

- Pekeris (1934), Markushevich (1994) – pair of adjoint Rayleigh Sturm-Liouville problems
- Beals, Henkin & Novikova (1995) – unphysical setting

setting

- for uniqueness: Jost function or spectral data at two distinct frequencies
- analysis for a finite (crust, upper mantle) slab beneath a traction-free surface (half space, flat earth)

Lamé parameters depend on the surface/boundary normal coordinate only inverse boundary value problem on a bounded, Lipschitz subdomain of $\mathbb{R}^3$

Nakamura & Uhlmann (1994) proved uniqueness assuming that the Lamé parameters are C^∞ and that the shear modulus is close to a positive constant

Eskin & Ralston (2002) proved a related result

Beretta, dH, Francini, Vessella & Zhai (2017) proved uniqueness and Lipschitz stability of such an inverse problem when the Lamé parameters and the density are assumed to be piecewise constant on a given domain partition

$$
\begin{aligned}
\frac{d}{dx}\left(\mu\frac{dw_1}{dx} - \xi\mu w_2\right) - \xi\lambda\frac{dw_2}{dx} + (\omega^2 - \xi^2(\lambda+2\mu))w_1 &= 0 \\
\frac{d}{dx}\left((\lambda+2\mu)\frac{dw_2}{dx} + \xi\lambda w_1\right) + \xi\mu\frac{dw_1}{dx} + (\omega^2 - \xi^2\mu)w_2 &= 0
\end{aligned}
$$

$x \in [0,\infty)$, supplemented with the (traction) boundary conditions

$$
\begin{aligned}
\left.\left(\mu\frac{dw_1}{dx} - \xi\mu w_2\right)\right|_{x=0^+} &= \chi_1 = 0 \\
\left.\left((\lambda+2\mu)\frac{dw_2}{dx} + \xi\lambda w_1\right)\right|_{x=0^+} &= \chi_2 = 0
\end{aligned}
$$

write $\chi = (\chi_1, \chi_2)^{\mathrm{T}}$ x is boundary normal coordinate

notation: use ξ for both $|\xi| \in \mathbb{R}_+$ and its values in $\mathbb{C}$ following analytic continuation

Lamé parameters normalized by density

Assumption

We let $\mu \geq \alpha_0 > 0$, $2\mu + 3\lambda \geq \beta_0 > 0$, $\lambda, \mu \in C^3(\mathbb{R}_+)$ *and* $\lambda(x) = \lambda_0$, $\mu(x) = \mu_0$ *for* $x \geq H$.

H signifies thickness of slab

Markushevich transform

let G be a 2×2-matrix solving the Cauchy problem,

$$G' = \frac{1}{2} L G, \quad G(0) = I_2$$

where I_2 is the unit matrix,

$$L = \begin{pmatrix} 0 & -d \\ -c & 0 \end{pmatrix} \quad \text{with} \quad c = \frac{1}{g_0} \frac{\mu(\lambda + \mu)}{(\lambda + 2\mu)}, \quad d = -2 g_0 \left(\frac{1}{\mu} \right)''$$

$\det G(x) = 1$

g_0 stands for an arbitrary positive constant; it is convenient to put $g_0 = \mu_0$

(inverse) Markushevich transform

$$\mathfrak{M}^{-1}(F) = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \quad \text{with} \quad \mathfrak{M}^{-1} = \begin{pmatrix} \dfrac{d}{dx} & 1 \\ -\xi & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\mu_0}{\mu} & 0 \\ 0 & \dfrac{\mu}{\lambda + 2\mu} \end{pmatrix} \left(G^{\mathrm{T}}\right)^{-1}$$

original system reduces to the matrix Sturm-Liouville form

$$\begin{aligned} F'' - \xi^2 F &= QF, \qquad x \in (0,\infty) \\ F' + \Theta F &= \left(D^{\mathrm{a}}\right)^{-1}\chi, \quad x = 0; \end{aligned} \qquad \Theta = \Theta(\xi) = (D^{\mathrm{a}}(\xi))^{-1} C^{\mathrm{a}}(\xi)$$

$$D^{\mathrm{a}}(\xi) = \begin{pmatrix} -2\mu_0 \dfrac{\mu'(0)}{\mu(0)} & \mu(0) \\ -2\mu_0\xi & 0 \end{pmatrix}, \; C^{\mathrm{a}}(\xi) = \begin{pmatrix} \mu_0\left(2\xi^2 - \dfrac{\omega^2}{\mu(0)} + \dfrac{\mu''(0)}{\mu(0)}\right) & -\dfrac{\mu'(0)\mu(0)}{\lambda(0) + 2\mu(0)} \\ 2\mu_0\xi\dfrac{\mu'(0)}{\mu(0)} & -\xi\dfrac{\mu^2(0)}{\lambda(0) + 2\mu(0)} \end{pmatrix}$$

Q is the matrix-valued potential: $Q = \left(G^{-1}BG\right)^{\mathrm{T}}, \quad B = B_1 + \omega^2 B_2$

$$(\mathfrak{M}^{\mathrm{a}})^{-1}(F^{\mathrm{a}}) = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \quad \text{with} \quad (\mathfrak{M}^{\mathrm{a}})^{-1} = \begin{pmatrix} 0 & -\xi \\ 1 & \dfrac{\mathrm{d}}{\mathrm{d}x} \end{pmatrix} \begin{pmatrix} 1 & -2\mu_0 \left(\dfrac{1}{\mu}\right)' \\ 0 & \dfrac{\mu_0}{\mu} \end{pmatrix} G$$

original system transforms to the matrix Sturm-Liouville form

$$\begin{aligned} (F^{\mathrm{a}})'' - \xi^2 F^{\mathrm{a}} &= Q^{\mathrm{a}} F^{\mathrm{a}}, \quad x \in (0,\infty) \\ (F^{\mathrm{a}})' + \Theta^{\mathrm{a}} F^{\mathrm{a}} &= D^{-1}\chi, \quad x = 0; \qquad Q^{\mathrm{a}} = Q^{\mathrm{T}}, \quad \Theta^{\mathrm{a}} = \Theta^{\mathrm{T}}(\xi) = D^{-1}(\xi) C(\xi) \end{aligned}$$

$$D(\xi) = \begin{pmatrix} 0 & -2\xi\mu_0 \\ \mu(0) & 0 \end{pmatrix}$$

$$C(\xi) = \begin{pmatrix} -\xi \dfrac{\mu^2(0)}{(\lambda(0) + 2\mu(0))} & 0 \\ -\mu'(0) & \dfrac{\mu_0}{\mu(0)} \left(2\mu(0)\xi^2 - \omega^2 - 2\dfrac{(\mu'(0))^2}{\mu(0)} + \mu''(0) \right) \end{pmatrix}$$

denote $Q(x)$ for $x \geq H$ by $Q_0(x)$ $\qquad c_0 = \frac{\lambda_0+\mu_0}{\lambda_0+2\mu_0}$, $G^H = G(H)$

$$Q_0(x) = \omega^2 \begin{pmatrix} -\dfrac{1}{\mu_0} & 0 \\ 0 & -\dfrac{1}{\lambda_0 + 2\mu_0} \end{pmatrix}$$

$$+ \omega^2 \frac{c_0}{\mu_0} \begin{pmatrix} -G^H_{12}\left[-\frac{c_0}{2} G^H_{11}(x-H) + G^H_{21}\right] & G^H_{11}\left[-\frac{c_0}{2} G^H_{11}(x-H) + G^H_{21}\right] \\ -G^H_{12}\left[-\frac{c_0}{2} G^H_{12}(x-H) + G^H_{22}\right] & G^H_{12}\left[-\frac{c_0}{2} G^H_{11}(x-H) + G^H_{21}\right] \end{pmatrix}$$

extend $Q_0 = Q_0(x)$ to $x \in (0, H]$, linear in x; $\qquad V(x) = Q(x) - Q_0(x)$, $V(x) = 0$ for $x \geq H$

Definition

A real matrix-valued potential, Q, is of Lamé type if it can be generated from Lamé parameters according to the Markushevich transform. Due to the Assumption, $Q \in C^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$.

boundary determination

The Lamé parameters at $x = 0$ and $x \geq H$, that is, $\lambda(0)$, $\mu(0)$ *as well as* $\mu'(0)$, $\mu''(0)$ *and* λ_0 *and* μ_0 *are encoded in, and determine* Θ *independently of* Q

we will not consider the problem of boundary determination

solutions to reference equations: $-F'' + Q_0 F = -\xi^2 F$ and $-(F^{\mathrm{a}})'' + Q_0^{\mathrm{T}} F^{\mathrm{a}} = -\xi^2 F^{\mathrm{a}}$

$$F_{P,0}^{\pm} = \begin{pmatrix} -\frac{c_0}{2} G_{11}^H (x-H) + G_{21}^H \pm \mathrm{i} q_P \frac{\mu_0}{\omega^2} G_{11}^H \\ -\frac{c_0}{2} G_{12}^H (x-H) + G_{22}^H \pm \mathrm{i} q_P \frac{\mu_0}{\omega^2} G_{12}^H \end{pmatrix} e^{\pm \mathrm{i} x q_P}$$

$$F_{S,0}^{\pm} = -\mu_0 \frac{\xi}{\omega^2} \begin{pmatrix} G_{11}^H \\ G_{12}^H \end{pmatrix} e^{\pm \mathrm{i} x q_S}$$

and

$$F_{S,0}^{\mathrm{a},\pm} = \begin{pmatrix} -\frac{c_0}{2} G_{12}^H (x-H) + G_{22}^H \mp \mathrm{i} q_S \frac{\mu_0}{\omega^2} G_{12}^H \\ \frac{c_0}{2} G_{11}^H (x-H) - G_{21}^H \pm \mathrm{i} q_S \frac{\mu_0}{\omega^2} G_{11}^H \end{pmatrix} e^{\pm \mathrm{i} x q_S}$$

$$F_{P,0}^{\mathrm{a},\pm} = \mu_0 \frac{\xi}{\omega^2} \begin{pmatrix} G_{12}^H \\ -G_{11}^H \end{pmatrix} e^{\pm \mathrm{i} x q_P}$$

quasi-momenta

$$q_P = \sqrt{\frac{\omega^2}{\lambda_0 + 2\mu_0} - \xi^2}$$

$$q_S = \sqrt{\frac{\omega^2}{\mu_0} - \xi^2}$$

cut complex plane

$$\mathcal{K} = \mathbb{C} \setminus \left(\left[-\frac{\omega}{\sqrt{\mu_0}}, \frac{\omega}{\sqrt{\mu_0}} \right] \cup \mathrm{i}\mathbb{R} \right)$$

Riemann surface

- Riemann surface $\mathcal{R}$ is obtained for both q_P and q_S by joining the separate Riemann surfaces for q_P and q_S so that q_P and q_S are single-valued holomorphic functions of ξ
- $\mathcal{R}$ is a four-fold cover of the plane; the part of $\mathcal{R}$ where $\operatorname{Im} q_P > 0$, $\operatorname{Im} q_S > 0$ is the physical ("upper") sheet $\mathcal{K}_+ = \mathcal{K}_{S,+}$
- $\zeta = \xi^2$ ("energies"); $\operatorname{Im} q_S(\zeta) > 0$, $\operatorname{Im} q_P(\zeta) > 0$ for $\zeta \in \Pi_+$,

$$\Pi_+ = \mathbb{C} \setminus \left(-\infty, \frac{\omega^2}{\mu_0}\right]$$

Jost solutions

Jost solutions, $F_P^{\pm}, F_S^{\pm}$ are determined by the conditions

$$F_P^{\pm} = F_{P,0}^{\pm}, \quad F_S^{\pm} = F_{S,0}^{\pm} \quad \text{for} \quad x \geq H$$

define the matrix Jost solution as

$$\mathbf{F}(x, \xi) = [F_P^{+} \ F_S^{+}]$$

and the Jost function (at the boundary, $x = 0$) as

$$\mathbf{F}_{\Theta}(\xi) = \mathbf{F}'(0, \xi) + \Theta(\xi)\mathbf{F}(0, \xi)$$

Jost solutions

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similarly for the adjoint problem

$$\mathbf{F}_{\Theta}^{\mathrm{a}}(\xi) = \begin{pmatrix} -2\frac{\mu_0}{\mu_0}\xi & 0 \\ \frac{\mu'(0)}{\mu(0)}\frac{1}{\xi} & -\frac{\mu(0)}{2\mu_0}\frac{1}{\xi} \end{pmatrix} \mathbf{F}_{\Theta}(\xi)$$

Weyl matrix

Weyl solution

$$\mathbf{\Phi}(x,\xi) = \mathbf{F}(x,\xi)[\mathbf{F}_\Theta(\xi)]^{-1}$$

Weyl matrix

$$\mathbf{M}(\xi) = \mathbf{\Phi}(0,\xi) = \mathbf{F}(0,\xi)[\mathbf{F}_\Theta(\xi)]^{-1}$$

$\mathbf{M}(\xi)\mathbf{F}_\Theta(\xi) = \mathbf{F}(0,\xi)$, whence $\mathbf{M}(\xi)$ can be identified with the *Robin-to-Dirichlet map* associated with the matrix Sturm-Liouville problem

Weyl solution

$$\mathbf{\Phi}(x,\xi) = \mathbf{F}(x,\xi)[\mathbf{F}_{\ominus}(\xi)]^{-1}$$

Weyl matrix

$$\mathbf{M}(\xi) = \mathbf{\Phi}(0,\xi) = \mathbf{F}(0,\xi)[\mathbf{F}_{\ominus}(\xi)]^{-1}$$

$\mathbf{M}(\xi)\mathbf{F}_{\ominus}(\xi) = \mathbf{F}(0,\xi)$, whence $\mathbf{M}(\xi)$ can be identified with the *Robin-to-Dirichlet map* associated with the matrix Sturm-Liouville problem

$$\mathbf{M}^{\mathrm{a}} = \mathbf{M}^{\mathrm{T}}$$

$\mathbf{M}$ (det) has a finite number (from asymptotics) of simple poles, at $\xi_1, \ldots, \xi_N$ (guided modes)

Assumption

The parameter functions, λ and μ, are such that there is no pole of $\mathbf{M}(\xi)$ with $\operatorname{Im} q_S = 0$ except, possibly, at $\xi = \frac{\omega}{\sqrt{\mu_0}}$ as a one-sided limit in $\mathcal{K}_+$.

Lemma

The matrix $\widehat{\mathbf{M}}$ admits the representation $\zeta_j = \xi_j^2$

$$\widehat{\mathbf{M}}(\zeta) = \int_{-\infty}^{\frac{\omega^2}{\mu_0}} \frac{\widehat{\mathbf{T}}(\eta)}{\zeta - \eta}\,\mathrm{d}\eta + \sum_{j=1}^{N} \frac{\alpha_j}{\zeta - \zeta_j}, \quad \zeta \in \Pi_+ \setminus \Lambda', \quad \Lambda' = \{\zeta_1, \ldots, \zeta_N\}$$

where

$$\alpha_j = \mathrm{Res}\,_{\zeta=\zeta_j} \widehat{\mathbf{M}}(\zeta) = \mathbf{F}(0, \xi_j) u_j, \quad u_j = 2\xi_j\,\mathrm{Res}\,_{\xi=\xi_j} [\mathbf{F}_\ominus(\xi)]^{-1}$$

or

$$\alpha_j = -[u_j^{\mathrm{a}}]^{\mathrm{T}} \int_0^\infty [\mathbf{F}^{\mathrm{a}}(x, \xi_j)]^{\mathrm{T}} \mathbf{F}(x, \xi)\,\mathrm{d}x\, u_j, \quad u_j^{\mathrm{a}} = 2\xi_j\,\mathrm{Res}\,_{\xi=\xi_j} [\mathbf{F}_\ominus^{\mathrm{a}}(\xi)]^{-1}$$

or

$$\alpha_j = \mathbf{F}(0, \xi_j) \left(\mathbf{F}_\ominus'(\xi_j)\right)^{-1} = -\mathrm{i}\frac{\mu_0}{\omega^2} \left[\left(\mathbf{F}_\ominus^{\mathrm{a}}(-\xi_j)\right)^{\mathrm{T}}\right]^{-1} \begin{pmatrix} q_P(\xi_j) & 0 \\ 0 & -q_S(\xi_j) \end{pmatrix} \left(\mathbf{F}_\ominus'(\xi_j)\right)^{-1}$$

and $\widehat{\mathbf{T}} = \widehat{\mathbf{T}}(\zeta)$, $\widehat{\mathbf{T}}(\zeta(\xi)) = \mathbf{T}(\xi)$ with

$$\mathbf{T}(\xi) = -\frac{\xi\mu_0}{\pi\omega^2}[(\mathbf{F}^{\mathrm{a}}_{\ominus})^{\mathrm{T}}(-\xi)]^{-1}\begin{pmatrix} q_P(\xi) & 0 \\ 0 & -q_S(\xi) \end{pmatrix}[\mathbf{F}_{\ominus}(\xi)]^{-1},$$

signifying the branch cut.

α_j and $\mathbf{T}$ can be expressed in terms of the Jost function only, thus the Lemma indicates that the Jost function encodes the boundary spectral data

- we assume that H, λ_0, μ_0, $\mu(0)$ and $\mu'(0)$ are known
- introduce the expansion of the Jost solution at the boundary

$$\mathbf{F}(0,\xi) = \xi \mathbf{G}_0(0,\xi) + \mathbf{G}_1(0) + \mathbf{R}(\xi), \quad \mathbf{R}(\xi) = \mathcal{O}\left(\frac{1}{|\xi|}\right)$$

we can construct explicit expressions for $\mathbf{G}_0(0,\xi)$ and $\mathbf{G}_1(0,\xi)$

Lemma

Given λ_0 and μ_0. The mapping from G^H to $(\mathbf{G}_0(0,\xi), \mathbf{G}_1(0,\xi))$ for any pair of frequencies, $\omega_1 \neq \omega_2 \in \mathbb{R}_+$, is an injection.

thus $(\mathbf{G}_0(0,\xi), \mathbf{G}_1(0,\xi))$ for any pair of frequencies, $\omega_1 \neq \omega_2$ determine G^H; moreover, G^H together with H, λ_0, μ_0 and ω determine Q_0

unique recovery

as λ_0, μ_0 are known, the Jost function determines the Weyl matrix

Proposition

Given G^H. For ω fixed, let V_1, V_2 be compactly supported on $[0, H]$ and belong to $L^1([0, H])$ with associated Weyl matrices $\mathbf{M}_1$, $\mathbf{M}_2$. If H, λ_0, μ_0, $\mu(0)$ and $\mu'(0)$ are known and the Assumptions hold true, then $\mathbf{M}_2(\xi) = \mathbf{M}_1(\xi)$ for all $\xi \in \mathcal{K}_+$ implies that $V_2 = V_1$.

proof: Gel'fand-Levitan-type equation, with some complications

thus, G^H together with $\mathbf{M}(\xi)$ determine V

unique recovery

by implication, $(\mathbf{G}_0(0,\xi), \mathbf{G}_1(0,\xi))$ for any two frequencies $\omega_1 \neq \omega_2 \in \mathbb{R}_+$ and $\mathbf{M}(\xi)$ determine Q

Theorem

Let Q_1, Q_2 be of Lamé type with associated Jost functions $\mathbf{F}_{\Theta;1}$, $\mathbf{F}_{\Theta;2}$. Assume that H, λ_0, μ_0, $\mu(0)$ and $\mu'(0)$ are known. Then $\mathbf{F}_{\Theta;2}(\xi) = \mathbf{F}_{\Theta;1}(\xi)$ for all $\xi \in \mathcal{K}_+$ and any pair of frequencies, $\omega_1 \neq \omega_2 \in \mathbb{R}_+$, subject to the Assumptions, implies that $Q_2 = Q_1$.

furthermore, from a Lamé-type Q for any pair of frequencies, $\omega_1 \neq \omega_2 \in \mathbb{R}_+$, one can recover λ and μ

unique recovery

by implication, $(\mathbf{G}_0(0,\xi), \mathbf{G}_1(0,\xi))$ for any two frequencies $\omega_1 \neq \omega_2 \in \mathbb{R}_+$ and $\mathbf{M}(\xi)$ determine Q

Theorem

Let Q_1, Q_2 be of Lamé type with associated Jost functions $\mathbf{F}_{\Theta;1}$, $\mathbf{F}_{\Theta;2}$. Assume that H, λ_0, μ_0, $\mu(0)$ and $\mu'(0)$ are known. Then $\mathbf{F}_{\Theta;2}(\xi) = \mathbf{F}_{\Theta;1}(\xi)$ for all $\xi \in \mathcal{K}_+$ and any pair of frequencies, $\omega_1 \neq \omega_2 \in \mathbb{R}_+$, subject to the Assumptions, implies that $Q_2 = Q_1$.

furthermore, from a Lamé-type Q for any pair of frequencies, $\omega_1 \neq \omega_2 \in \mathbb{R}_+$, one can recover λ and μ *reconciling seismology with analysis*

- we need both the Weyl matrix and the Jost solution at the boundary for the unique recovery of Lamé parameters
- assuming that λ_0 and μ_0 are known, the Jost function determines the Weyl matrix and the Jost solution at the boundary

(Love and) Rayleigh resonances

leaky modes: Rayleigh resonances

- Rosenbaum (1960)
- Phinney (1961) – theoretical study of leaky waves, referred to as pseudo-P modes
- Haddon (1986) – evaluation of the response of a layered elastic medium to an explosive point source ($\sim$ resolvent) using leaking modes
- Schröder & Scott (2001) – study of complex conjugate roots of the Rayleigh equation
- García-Jerez & Sánchez-Sesma (2014) – P-SV leaky waves
- Gao, Xia & Pan (2014)

return to Rayleigh system

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}Z}\Big(\mu\frac{\mathrm{d}w_1}{\mathrm{d}Z}\Big) + \mathrm{i}\xi\Big(\frac{\mathrm{d}}{\mathrm{d}Z}(\mu w_2) + \lambda\frac{\mathrm{d}w_2}{\mathrm{d}Z}\Big) + \big(\omega^2 - \xi^2(\lambda+2\mu)\big)w_1 &= 0\\
\frac{\mathrm{d}}{\mathrm{d}Z}\Big((\lambda+2\mu)\frac{\mathrm{d}w_2}{\mathrm{d}Z}\Big) + \mathrm{i}\xi\Big(\frac{\mathrm{d}}{\mathrm{d}Z}(\lambda w_1 + \mu\frac{\mathrm{d}w_1}{\mathrm{d}Z}\Big) + (\omega^2 - \xi^2\mu)w_2 &= 0
\end{aligned}$$

$Z \in (-\infty, 0]$, supplemented with the (traction) boundary conditions

$$\begin{aligned}
\chi_1 = \Big(\mu\frac{\mathrm{d}w_1}{\mathrm{d}Z} + \mathrm{i}\xi\mu w_2\Big)\Big|_{Z=0^-} &=: \quad a(w) = 0\\
\chi_2 = \Big((\lambda+2\mu)\frac{\mathrm{d}w_2}{\mathrm{d}Z} + \mathrm{i}\xi\lambda w_1\Big)\Big|_{Z=0^-} &=: \quad b(w) = 0
\end{aligned}$$

Z is boundary normal coordinate

notation: use ξ for both $|\xi| \in \mathbb{R}_+$ and its values in $\mathbb{C}$ following analytic continuation

Lamé parameters normalized by density

Assumption

We let $\mu \geq \alpha_0 > 0$, $2\mu + 3\lambda \geq \beta_0 > 0$, $\lambda, \mu \in C^3(\mathbb{R}_-)$; $\lambda(Z) = \lambda_0$, $\mu(Z) = \mu_0$ *for* $Z \leq -H$.

H signifies thickness of slab, $Z_I := -H$

quasi-momenta

$$
\begin{aligned}
q_P &= \sqrt{\frac{\omega^2}{\lambda_0 + 2\mu_0} - \xi^2} \\
q_S &= \sqrt{\frac{\omega^2}{\mu_0} - \xi^2}
\end{aligned}
$$

Riemann surface

- Riemann surface $\mathcal{R}$ is obtained for both q_P and q_S by joining the separate Riemann surfaces for q_P and q_S so that q_P and q_S are single-valued holomorphic functions of ξ
- $\mathcal{R}$ is a four-fold cover of the plane; the sheets of $\mathcal{R}$,

$$\mathcal{R} = \mathcal{R}_{++} \cup \mathcal{R}_{+-} \cup \mathcal{R}_{-+} \cup \mathcal{R}_{--} = \cup_{\sigma_1,\sigma_2} \mathcal{R}_{\sigma_1,\sigma_2}, \quad (\sigma_1, \sigma_2) = (\operatorname{sign} \operatorname{Im} q_P, \operatorname{sign} \operatorname{Im} q_S)$$

to a point $\xi \in \mathcal{R}$ we may associate the two values $q_S(\xi)$, $q_P(\xi)$ and can determine a mapping $\mathcal{R} \to \mathcal{R}$ by its action on $q_S(\xi)$, $q_P(\xi)$; thus, we define mappings, w_P, w_S and w_{SP} : $\mathcal{R} \to \mathcal{R}$

$$\begin{aligned} q_S(w_S(\xi)) &= -q_S(\xi), \quad q_P(w_S(\xi)) = q_P(\xi) \\ q_S(w_P(\xi)) &= q_S(\xi), \quad q_P(w_P(\xi)) = -q_P(\xi) \\ q_S(w_{SP}(\xi)) &= -q_S(\xi), \quad q_P(w_{SP}(\xi)) = -q_P(\xi) \end{aligned}$$

relations, between the sheets of the Riemann surface, map a point $\xi \in \mathcal{R}$ to another point in $\mathcal{R}$

Jost solutions $f_P^\pm$, $f_S^\pm$ for $Z < 0$ satisfy the conditions

$$f_P^\pm = f_{P,0}^\pm, \qquad f_S^\pm = f_{S,0}^\pm \quad \text{for } Z < Z_I$$

where

$$f_P^\pm = \begin{pmatrix} f_{P,1}^\pm \\ f_{P,2}^\pm \end{pmatrix} = \begin{pmatrix} \xi \\ \pm q_P \end{pmatrix} e^{\pm \mathrm{i} Z q_P}, \quad Z < Z_I$$

$$f_S^\pm = \begin{pmatrix} f_{S,1}^\pm \\ f_{S,2}^\pm \end{pmatrix} = \begin{pmatrix} \pm q_S \\ -\xi \end{pmatrix} e^{\pm \mathrm{i} Z q_S}, \quad Z < Z_I$$

extend $\mu(Z)$, $\lambda(Z)$ as even functions to $Z > 0$; with these, extend the system to the real line

by abuse of notation, we use the same notation, $f_P^\pm$, $f_S^\pm$, for the Jost solutions satisfying the evenly extended system

boundary matrix

denote

$$\mathscr{B} = \begin{pmatrix} a(f_P^-) & a(f_S^-) \\ b(f_P^-) & b(f_S^-) \end{pmatrix}, \quad \mathscr{B} = \mathscr{B}(\xi)$$

signifying the *boundary matrix* representing boundary tractions induced by the Jost solutions

the boundary matrix determines the Jost function via the inverse Markushevich transform assuming that $\mu(0)$, $\mu'(0)$ and μ_0 are known

Rayleigh determinant

$$\Delta = \det \mathscr{B}$$

decomposition into entire functions

define

$$\vartheta_P = \frac{1}{2}\left(f_P^+ + f_P^-\right), \quad \varphi_P = \frac{1}{2q_P}\left(f_P^+ - f_P^-\right), \quad \vartheta_S = \frac{1}{2}\left(f_S^+ + f_S^-\right), \quad \varphi_S = \frac{1}{2q_S}\left(f_S^+ - f_S^-\right)$$

boundary matrix takes the form

$$\mathscr{B} = \begin{pmatrix} a(\vartheta_P) & a(\vartheta_S) \\ b(\vartheta_P) & b(\vartheta_S) \end{pmatrix} - \begin{pmatrix} a(\varphi_P) & a(\varphi_S) \\ b(\varphi_P) & b(\varphi_S) \end{pmatrix} \begin{pmatrix} q_P & 0 \\ 0 & q_S \end{pmatrix}$$

Rayleigh determinant takes the form

$$\Delta = d_1 + q_P d_2 + q_S d_3 + q_P q_S d_4$$

where

$$d_1 = \det \begin{pmatrix} a(\vartheta_P) & a(\vartheta_S) \\ b(\vartheta_P) & b(\vartheta_S) \end{pmatrix}, \; d_2 = -\det \begin{pmatrix} a(\varphi_P) & a(\vartheta_S) \\ b(\varphi_P) & b(\vartheta_S) \end{pmatrix}$$

$$d_3 = -\det \begin{pmatrix} a(\vartheta_P) & a(\varphi_S) \\ b(\vartheta_P) & b(\varphi_S) \end{pmatrix}, \; d_4 = \det \begin{pmatrix} a(\varphi_P) & a(\varphi_S) \\ b(\varphi_P) & b(\varphi_S) \end{pmatrix};$$

decomposition into entire functions

define

$$\vartheta_P = \frac{1}{2}\left(f_P^+ + f_P^-\right), \quad \varphi_P = \frac{1}{2q_P}\left(f_P^+ - f_P^-\right), \quad \vartheta_S = \frac{1}{2}\left(f_S^+ + f_S^-\right), \quad \varphi_S = \frac{1}{2q_S}\left(f_S^+ - f_S^-\right)$$

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intermediate function, Rayleigh resonances

$$F(\xi) = \Delta(\xi)\Delta(w_S(\xi))\Delta(w_P(\xi))\Delta(w_{PS}(\xi)).$$

is in a Cartwright class ($\mathbb{C}_{4H}$)

Rayleigh resonance "frequencies" are the zeros of the Rayleigh determinant; they are grouped in sets

$$\Sigma_{++},\ \Sigma_{+-},\ \Sigma_{-+},\ \Sigma_{--}$$

on the four sheets, $\mathcal{R}_{++}$, $\mathcal{R}_{+-}$, $\mathcal{R}_{-+}$, $\mathcal{R}_{--}$; that is,

$$\begin{array}{rcll} \Delta(\xi_j) & = & 0, & \xi_j \in \Sigma_{++}, \\ \Delta(w_P(\xi_j)) & = & 0, & \xi_j \in \Sigma_{-+}, \\ \Delta(w_S(\xi_j)) & = & 0, & \xi_j \in \Sigma_{+-}, \\ \Delta(w_{PS}(\xi_j)) & = & 0, & \xi_j \in \Sigma_{--} \end{array}$$

the set Σ_{++} corresponds with Regge "bound states"

- F can be recovered from resonance frequencies (Hadamard factorization)
- $\mathcal{S}$ can be recovered from "frequencies" at which no mode conversion occurs

Conjecture

The boundary matrix can be recovered from the resonance frequencies and $\mathcal{S}$.

recovery follows from applying the theorem for spectral data

Lemma

On the Riemann surface $\mathcal{R}$, the following holds true

$$f_P^{\pm}(Z, w_P(\xi)) = f_P^{\pm}(Z, w_{PS}(\xi)) = f_P^{\mp}(Z, \xi),$$
$$f_S^{\pm}(Z, w_S(\xi)) = f_S^{\pm}(Z, w_{PS}(\xi)) = f_S^{\mp}(Z, \xi).$$

physical sheet

identify $\mathcal{R}_{++}$ where $\mathrm{Im}\, q_P > 0$, $\mathrm{Im}\, q_S > 0$ with the physical (or "upper") sheet for q_S

$$\mathcal{K}_{++} = \left\{ \xi \in \mathcal{K}_S = \mathbb{C} \setminus \left(\left[-\frac{\omega}{\sqrt{\mu_0}}, \frac{\omega}{\sqrt{\mu_0}} \right] \cup \mathrm{i}\mathbb{R} \right) \ : \ \mathrm{Re}\, \xi > 0 \right\}$$

on $\mathcal{K}_{++}$ we have $\mathrm{Im}\, q_P > \mathrm{Im}\, q_S$