Seismic normal modes, Rayleigh waves, resonances and inverse problems
– reconciliation of seismology with analysis

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terrestrial planets, discrete spectrum

omitting rotation

seismic normal modes

decomposition of natural Hilbert space

normal modes, Raleigh waves, resonances
geometrical setup \(- c = c_P, c_S\)

radial manifold with boundary, \(M = \overline{B(0,1)}\) – Riemannian metric

\[ g(x) = c^{-2}(|x|)e(x), \quad c: (0, 1] \rightarrow (0, \infty) \]

e is the standard Euclidean metric

\[ c(r) \] has a jump discontinuity at a finite set of values \( r = r_1, \cdots, r_K \); that is

\[ \lim_{r \rightarrow r_k^-} c(r) \neq \lim_{r \rightarrow r_k^+} c(r) \] for each \( i \) (annuli \( A(r_{k-1}, r_k) \))

a maximal geodesic is a unit speed geodesic on the Riemannian manifold with each endpoint at its boundary or at an interface

a broken ray is a concatenation of maximal geodesics satisfying the reflection condition of geometrical optics at both inner and outer boundaries of \( M \), and Snell’s law for geometric optics at the interfaces
Conditions

Herglotz condition

$$\frac{d}{dr} \left( \frac{r}{c(r)} \right) > 0$$

Away from discontinuities
A broken ray is called basic if

- it stays within a single layer and all of its legs are reflections from a single interface, or
- it is a radial ray contained in a single layer; such a ray is defined to be a ray with zero epicentral distance and will necessarily reflect from two interfaces.

Let $\gamma$ be a basic ray with radius $R^*$ ($r_k \leq R^* < r_{k+1}$), (conserved) ray parameter $p$, which lies inside $A(r_{k-1}, r_k)$ ($1 = r_0 > r_1 > \cdots > r_K$); there is a unique $N \in \mathbb{N}$ so that its length $T$ is

$$T = 2NL\gamma := 2N \int_{R^*}^{r_{k-1}} \frac{1}{c(r')^2 \beta(r'; p)} \, dr', \quad \beta(r; p)^2 = c(r)^{-2} - r^{-2} p^2$$

and angular or epicentral distance

$$\alpha_\gamma := \alpha(p) = 2N \int_{R^*}^{r_{k-1}} \frac{p}{(r')^2 \beta(r'; p)} \, dr'$$
Consider geodesics in an annulus \( A(a, b) \) equipped with a \( C^{1,1} \) wave speed \( c : (a, b] \to (0, \infty) \). It satisfies the \textit{countable conjugacy condition} if there are only countably many radii \( r \in (a, b) \) so that the endpoints of the corresponding maximal geodesic \( \gamma(r) \) are conjugate along that geodesic.

The radial wave speed \( c \) satisfies the \textit{periodic conjugacy condition} if for each periodic, nongliding ray with a ray parameter \( p \), \( \partial_p \alpha(p) \neq 0 \). (This ensures that the phase function in the stationary phase argument for computing the trace formula is Bott-Morse nondegenerate.)

\[ c_\tau : [0, 1] \to (0, \infty) \text{ indexed by } \tau \in (-\varepsilon, \varepsilon) \text{ is an “admissible” family of profiles} \]
(basic) length spectrum

Length spectrum, $\text{lsp}(c)$: the set of lengths of all periodic broken rays

Basic length spectrum: $\text{blsp}(c)$

$P$ in blue, $S$ in red (PKPab, PKIKP, SP, SKKS, PKJKP)
conditions

- equivalence classes \([\gamma]\) (rotations, time reversal, dynamic analogs) parameterized by \(p\)
- \(Q_{[\gamma]}\) is product of reflection and transmission coefficients (transmission conditions)
- \(n_{[\gamma]}\) is number of dynamic analogs

**Definition**

The length spectrum satisfies the *principal amplitude injectivity condition* if given two closed rays \(\gamma_1\) and \(\gamma_2\) with the same period and disjoint equivalence classes (so they must have different ray parameters \(p_1\) and \(p_2\), then

\[
n_{[\gamma_1]} Q_{[\gamma_1]} |p_1^{-2} \partial_p \alpha(p_1)|^{-1/2} \neq n_{[\gamma_2]} Q_{[\gamma_2]} |p_2^{-2} \partial_p \alpha(p_2)|^{-1/2}
\]

ensuring recovery of \(T\).
Theorem

Fix any $\varepsilon > 0$ and $K \in \mathbb{N}$, and let $c_\tau(r)$ be an admissible family of profiles with discontinuities at $r_k(\tau)$ for all $k = 1, \ldots, K$. Let $\text{blsp}(\tau)$ denote the basic length spectrum with the wave speed profile $c_\tau$. Suppose $\text{blsp}(\tau)$ is countable for all $\tau$. Let $S(\tau)$ be any collection of countable subsets of $\mathbb{R}$ indexed by $\tau$.

If $\text{blsp}(\tau) \cup S(\tau) = \text{blsp}(0) \cup S(0)$ for all $\tau \in (-\varepsilon, \varepsilon)$, then $c_\tau = c_0$ and $r_k(\tau) = r_k(0)$ for all $\tau \in (-\varepsilon, \varepsilon)$ and $k = 1, \ldots, K$. 
Corollary (Length spectral rigidity with two polarizations)

Fix any $\varepsilon > 0$ and $K \in \mathbb{N}$, and let $c^i_\tau(r)$ with both $i = 1, 2$ be an admissible family of profiles with discontinuities at $r_k(\tau)$ for all $k = 1, \ldots, K$. Consider all periodic rays which are geodesics within each layer and satisfy the usual reflection or transmission conditions at interfaces, but which can change between the wave speed profiles $c^1_\tau$ and $c^2_\tau$ at any reflection and transmission. Suppose that the length spectrum of this whole family of geodesics, denoted by $\text{lsp}(\tau)$, is countable in the ball $B(0, 1)$.

If $\text{lsp}(\tau) = \text{lsp}(0)$ for all $\tau \in (-\varepsilon, \varepsilon)$, then $c^i_\tau = c^i_0$ for both $i = 1, 2$ and $r_k(\tau) = r_k(0)$ for all $\tau \in (-\varepsilon, \varepsilon)$ and $k = 1, \ldots, K$. 

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Theorem (Spectral rigidity with moving interfaces)

Fix any $\varepsilon > 0$ and $K \in \mathbb{N}$, and let $c_\tau(r)$ be an admissible family of profiles with discontinuities at $r_k(\tau)$ for all $k = 1, \ldots, K$. Suppose that the length spectrum for each $c_\tau$ is countable in the ball $\bar{B}(0, 1) \subset \mathbb{R}^3$. Assume also that the length spectrum satisfies the principal amplitude injectivity condition and the periodic conjugacy condition.

Suppose $\text{spec}(\tau) = \text{spec}(0)$ for all $\tau \in (-\varepsilon, \varepsilon)$. Then $c_\tau = c_0$ and $r_k(\tau) = r_k(0)$ for all $\tau \in (-\varepsilon, \varepsilon)$ and $k = 1, \ldots, K$.

trace formula – possible periodic broken rays $\gamma_0$, say, with gliding

- gliding occurs at only one interface; this is ensured by the Herglotz condition
- there is a sequence of periodic non-gliding broken rays $\gamma_i$ so that $\gamma_i \to \gamma_0$; subtlety lies in ensuring periodicity of the approximating rays
“near” phase boundaries, Earth’s surface

Love and Rayleigh waves: local recovery

Stoneley → Rayleigh
Earth as a unit ball $B_1 = B(0, 1)$; there is a global diffeomorphism, $\phi$

$$\phi : B_1 \setminus \{0\} \to S^2 \times \mathbb{R}$$

$$\phi(B_r) = S^2 \times \left\{1 - \frac{1}{r}\right\}, \ r \neq 0$$

- for an open and bounded subset $U \subset S^2$, the cone region, $\{(\Theta, r) \mid \Theta \in U, \ 0 < r < 1\}$, is diffeomorphic to $U \times \mathbb{R}^{-}$; we can find global coordinates for $U$ and we may consider our system on the domain $S^2 \times \mathbb{R}^{-}$
- more generally, we consider the system on any Riemannian manifold of the form $M = \partial M \times \mathbb{R}^{-}$ with metric

$$g = \begin{pmatrix} g' & 0 \\ 0 & 1 \end{pmatrix}$$

- for a “nice” domain $\Omega$, a neighborhood of the boundary is diffeomorphic to $M$, where the metric $g'$ is the induced metric of the boundary of $\Omega$
Rayleigh waves/modes have long/widely been used to study Earth’s crust and upper mantle (Dorman & Ewing, 1962).

Empirically it has been established that “phase velocities” or eigenvalues (fundamental mode and overtones) at a few discrete frequencies are insufficient data to determine both $P$- and $S$-wave speeds (Lamé parameters).

It is now common practice to add data: “H/V” related to the components of the trace of modes, and information from body waves/modes.


setting

- for uniqueness: Jost function or spectral data at two distinct frequencies
- analysis for a finite (crust, upper mantle) slab beneath a traction-free surface (half space, flat earth)

Lamé parameters depend on the surface/boundary normal coordinate only
some history

inverse boundary value problem on a bounded, Lipschitz subdomain of $\mathbb{R}^3$

Nakamura & Uhlmann (1994) proved uniqueness assuming that the Lamé parameters are $C^\infty$ and that the shear modulus is close to a positive constant.

Eskin & Ralston (2002) proved a related result.

Beretta, dH, Francini, Vessella & Zhai (2017) proved uniqueness and Lipschitz stability of such an inverse problem when the Lamé parameters and the density are assumed to be piecewise constant on a given domain partition.
Rayleigh system

\[
\frac{d}{dx}\left(\mu \frac{dw_1}{dx} - \xi \mu w_2\right) - \xi \lambda \frac{dw_2}{dx} + (\omega^2 - \xi^2(\lambda + 2\mu)) w_1 = 0
\]

\[
\frac{d}{dx}\left((\lambda + 2\mu) \frac{dw_2}{dx} + \xi \lambda w_1\right) + \xi \mu \frac{dw_1}{dx} + (\omega^2 - \xi^2 \mu) w_2 = 0
\]

\(x \in [0, \infty),\) supplemented with the (traction) boundary conditions

\[
\left.\left(\mu \frac{dw_1}{dx} - \xi \mu w_2\right)\right|_{x=0^+} = \chi_1 = 0
\]

\[
\left.\left((\lambda + 2\mu) \frac{dw_2}{dx} + \xi \lambda w_1\right)\right|_{x=0^+} = \chi_2 = 0
\]

write \(\chi = (\chi_1, \chi_2)^T\)

\(x\) is boundary normal coordinate

notation: use \(\xi\) for both \(|\xi| \in \mathbb{R}_+\) and its values in \(\mathbb{C}\) following analytic continuation
Lamé parameters normalized by density

**Assumption**

We let \( \mu \geq \alpha_0 > 0, 2\mu + 3\lambda \geq \beta_0 > 0, \lambda, \mu \in C^3(\mathbb{R}^+) \) and \( \lambda(x) = \lambda_0, \mu(x) = \mu_0 \) for \( x \geq H \).

\( H \) signifies thickness of slab
Markushevich transform

let $G$ be a $2 \times 2$-matrix solving the Cauchy problem,

$$G' = \frac{1}{2} LG, \quad G(0) = I_2$$

where $I_2$ is the unit matrix,

$$L = \begin{pmatrix} 0 & -d \\ -c & 0 \end{pmatrix} \quad \text{with} \quad c = \frac{1}{g_0} \frac{\mu(\lambda + \mu)}{(\lambda + 2\mu)}, \quad d = -2g_0 \left( \frac{1}{\mu} \right)''$$

$$\det G(x) = 1$$

$g_0$ stands for an arbitrary positive constant; it is convenient to put $g_0 = \mu_0$
(inverse) Markushevich transform

\[
\mathcal{M}^{-1}(F) = \begin{pmatrix} \frac{w_1}{w_2} \end{pmatrix} \quad \text{with} \quad \mathcal{M}^{-1} = \begin{pmatrix} \frac{d}{d\chi} & 1 & \mu_0 & 0 \\ -\xi & 0 & \mu & \lambda + 2\mu \end{pmatrix} \begin{pmatrix} 0 \\ \mu \end{pmatrix}, \quad (G^T)^{-1}
\]

original system reduces to the matrix Sturm-Liouville form

\[
F'' - \xi^2 F = QF, \quad x \in (0, \infty)
\]
\[
F' + \Theta F = (D^a)^{-1} \chi, \quad x = 0; \quad \Theta = \Theta(\xi) = (D^a(\xi))^{-1} C^a(\xi)
\]

\[
D^a(\xi) = \begin{pmatrix} -2\mu_0 \frac{\mu'(0)}{\mu(0)} & \frac{\mu(0)}{0} \\ -2\mu_0 \xi & 0 \end{pmatrix}, \quad C^a(\xi) = \begin{pmatrix} \mu_0 \left( 2\xi^2 - \frac{\omega^2}{\mu(0)} + \frac{\mu''(0)}{\mu(0)} \right) - \frac{\mu'(0)\mu(0)}{\lambda(0) + 2\mu(0)} \\ 2\mu_0 \xi \frac{\mu'(0)}{\mu(0)} - \xi \frac{\mu^2(0)}{\lambda(0) + 2\mu(0)} \end{pmatrix}
\]

\[Q\] is the matrix-valued potential: \[Q = (G^{-1}BG)^T, \quad B = B_1 + \omega^2 B_2\]
adjoint problem

\[(\mathcal{M}^a)^{-1}(F^a) = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \quad \text{with} \quad (\mathcal{M}^a)^{-1} = \begin{pmatrix} 0 & -\xi \\ 1 & \frac{d}{dx} \end{pmatrix} \begin{pmatrix} 1 & -2\mu_0 \left(\frac{1}{\mu}\right)' \\ 0 & \frac{\mu_0}{\mu} \end{pmatrix} \]

original system transforms to the matrix Sturm-Liouville form

\[(F^a)'' - \xi^2 F^a = Q^a F^a, \quad x \in (0, \infty)\]

\[(F^a)' + \Theta^a F^a = D^{-1} \chi, \quad x = 0; \quad Q^a = Q^T, \quad \Theta^a = \Theta^T(\xi) = D^{-1}(\xi) C(\xi)\]

\[D(\xi) = \begin{pmatrix} 0 & -2\xi\mu_0 \\ \mu(0) & 0 \end{pmatrix}\]

\[C(\xi) = \begin{pmatrix} \frac{\mu^2(0)}{(\lambda(0) + 2\mu(0))} & 0 \\ -\mu'(0) & \frac{\mu_0}{\mu(0)} \left(2\mu(0)\xi^2 - \omega^2 - 2\left(\frac{\mu'(0)}{\mu(0)}\right)^2 + \mu''(0)\right) \end{pmatrix}\]
denote \( Q(x) \) for \( x \geq H \) by \( Q_0(x) \)

\[
Q_0(x) = \omega^2 \begin{pmatrix}
-\frac{1}{\mu_0} & 0 \\
0 & -\frac{1}{\lambda_0 + 2\mu_0}
\end{pmatrix}
\]

\[
+ \omega^2 \frac{c_0}{\mu_0} \begin{pmatrix}
-G_{12}^H \left[-\frac{c_0}{2} G_{11}^H(x - H) + G_{21}^H\right] & G_{11}^H \left[-\frac{c_0}{2} G_{11}^H(x - H) + G_{21}^H\right] \\
-G_{12}^H \left[-\frac{c_0}{2} G_{12}^H(x - H) + G_{22}^H\right] & G_{12}^H \left[-\frac{c_0}{2} G_{11}^H(x - H) + G_{21}^H\right]
\end{pmatrix}
\]

extend \( Q_0 = Q_0(x) \) to \( x \in (0, H] \), linear in \( x \);

\[
V(x) = Q(x) - Q_0(x), \quad V(x) = 0 \text{ for } x \geq H
\]

**Definition**

A real matrix-valued potential, \( Q \), is of Lamé type if it can be generated from Lamé parameters according to the Markushevich transform. Due to the Assumption, \( Q \in C^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+) \).
boundary determination

The Lamé parameters at $x = 0$ and $x \geq H$, that is, $\lambda(0)$, $\mu(0)$ as well as $\mu'(0)$, $\mu''(0)$ and $\lambda_0$ and $\mu_0$ are encoded in, and determine $\Theta$ independently of $Q$

we will not consider the problem of boundary determination
solutions to reference equations: \(-F'' + Q_0 F = -\xi^2 F\) and \(-(F^a)' + Q_0^T F^a = -\xi^2 F^a\)

\[
F_{P,0}^\pm = \left( \begin{array}{c}
-\frac{c_0}{2} G_{11}^H(x - H) + G_{21}^H \pm i q_P \frac{\mu_0}{\omega^2} G_{11}^H \\
-\frac{c_0}{2} G_{12}^H(x - H) + G_{22}^H \pm i q_P \frac{\mu_0}{\omega^2} G_{12}^H 
\end{array} \right) e^{\pm i x q_P}
\]

\[
F_{S,0}^\pm = -\mu_0 \frac{\xi}{\omega^2} \begin{pmatrix} G_{11}^H \\ G_{12}^H \end{pmatrix} e^{\pm i x q_S}
\]

and

\[
F_{S,0}^{a,\pm} = \left( \begin{array}{c}
-\frac{c_0}{2} G_{12}^H(x - H) + G_{22}^H \mp i q_S \frac{\mu_0}{\omega^2} G_{12}^H \\
\frac{c_0}{2} G_{11}^H(x - H) - G_{21}^H \mp i q_S \frac{\mu_0}{\omega^2} G_{11}^H 
\end{array} \right) e^{\pm i x q_S}
\]

\[
F_{P,0}^{a,\pm} = \mu_0 \frac{\xi}{\omega^2} \begin{pmatrix} G_{12}^H \\ -G_{11}^H \end{pmatrix} e^{\pm i x q_P}
\]

quasi-momenta

\[
q_P = \sqrt{\frac{\omega^2}{\lambda_0 + 2\mu_0} - \xi^2}
\]

\[
q_S = \sqrt{\frac{\omega^2}{\mu_0} - \xi^2}
\]

cut complex plane

\[
\mathcal{K} = \mathbb{C} \setminus \left( \left[ -\frac{\omega}{\sqrt{\mu_0}}, \frac{\omega}{\sqrt{\mu_0}} \right] \cup i\mathbb{R} \right)
\]
• Riemann surface \( \mathcal{R} \) is obtained for both \( q_P \) and \( q_S \) by joining the separate Riemann surfaces for \( q_P \) and \( q_S \) so that \( q_P \) and \( q_S \) are single-valued holomorphic functions of \( \xi \).

• \( \mathcal{R} \) is a four-fold cover of the plane; the part of \( \mathcal{R} \) where \( \text{Im} \ q_P > 0, \text{Im} \ q_S > 0 \) is the physical (“upper”) sheet \( \mathcal{K}_+ = \mathcal{K}_{S,+} \).

• \( \zeta = \xi^2 \) (“energies”); \( \text{Im} \ q_S(\zeta) > 0, \text{Im} \ q_P(\zeta) > 0 \) for \( \zeta \in \Pi_+ \),

\[
\Pi_+ = \mathbb{C} \setminus \left( -\infty, \frac{\omega^2}{\mu_0} \right]
\]
Jost solutions

Jost solutions, $F_P^\pm, F_S^\pm$ are determined by the conditions

$$F_P^\pm = F_P^\pm, \quad F_S^\pm = F_S^\pm$$

for $x \geq H$

define the matrix Jost solution as

$$F(x, \xi) = [F_P^+, F_S^+]$$

and the Jost function (at the boundary, $x = 0$) as

$$F_\Theta(\xi) = F'(0, \xi) + \Theta(\xi)F(0, \xi)$$
Jost solutions

Jost solutions, $F^\pm_P$, $F^\pm_S$ are determined by the conditions

$$F^\pm_P = F^\pm_P, \quad F^\pm_S = F^\pm_S, \quad \text{for} \quad x \geq H$$

define the matrix Jost solution as

$$F(x, \xi) = \begin{bmatrix} F^+_P & F^+_S \end{bmatrix}$$

and the Jost function (at the boundary, $x = 0$) as

$$F_{\Theta}(\xi) = F'(0, \xi) + \Theta(\xi)F(0, \xi)$$

similarly for the adjoint problem

$$F^a_{\Theta}(\xi) = \begin{pmatrix} -2\frac{\mu_0}{\xi} & 0 \\ \frac{\mu'(0)}{\mu(0)} & -\frac{\mu(0)}{2\mu_0} \xi \end{pmatrix} F_{\Theta}(\xi)$$
Weyl solution

\[ \Phi(x, \xi) = F(x, \xi)[F_\Theta(\xi)]^{-1} \]

Weyl matrix

\[ M(\xi) = \Phi(0, \xi) = F(0, \xi)[F_\Theta(\xi)]^{-1} \]

\[ M(\xi)F_\Theta(\xi) = F(0, \xi), \text{ whence } M(\xi) \text{ can be identified with the Robin-to-Dirichlet map} \]

associated with the matrix Sturm-Liouville problem
Weyl matrix

Weyl solution
\[ \Phi(x, \xi) = F(x, \xi)[F_\Theta(\xi)]^{-1} \]

Weyl matrix
\[ M(\xi) = \Phi(0, \xi) = F(0, \xi)[F_\Theta(\xi)]^{-1} \]

\[ M(\xi)F_\Theta(\xi) = F(0, \xi), \text{ whence } M(\xi) \text{ can be identified with the } \textit{Robin-to-Dirichlet map} \]

associated with the matrix Sturm-Liouville problem

\[ M^a = M^T \]

\( M \) (det) has a finite number (from asymptotics) of simple poles, at \( \xi_1, \ldots, \xi_N \) (guided modes)

Assumption

The parameter functions, \( \lambda \) and \( \mu \), are such that there is no pole of \( M(\xi) \) with \( \text{Im } q_S = 0 \) except, possibly, at \( \xi = \frac{\omega}{\sqrt{\mu_0}} \) as a one-sided limit in \( K_+ \).
\[ \hat{M}(\zeta(\xi)) = M(\xi), \zeta(\xi) = \xi^2 \]

**Lemma**

The matrix \( \hat{M} \) admits the representation

\[
\hat{M}(\zeta) = \int_{-\infty}^{\infty} \frac{\hat{T}(\eta)}{\zeta - \eta} \, d\eta + \sum_{j=1}^{N} \frac{\alpha_j}{\zeta - \zeta_j}, \quad \zeta \in \Pi_+ \setminus \Lambda', \quad \Lambda' = \{\zeta_1, \ldots, \zeta_N\}
\]

where

\[
\alpha_j = \text{Res}_{\xi=\zeta_j} \hat{M}(\zeta) = F(0, \xi_j) u_j, \quad u_j = 2\xi_j \text{Res}_{\xi=\xi_j} [F_{\Theta}(\xi)]^{-1}
\]

or

\[
\alpha_j = -[u_j^a]^T \int_0^{\infty} [F^a(x, \xi_j)]^T F(x, \xi) \, dx \, u_j, \quad u_j^a = 2\xi_j \text{Res}_{\xi=\xi_j} [F_{\Theta}^a(\xi)]^{-1}
\]

or

\[
\alpha_j = F(0, \xi_j) (F'_{\Theta}(\xi_j))^{-1} = -i\frac{\mu_0}{\omega^2} \left[ (F^a_{\Theta}(-\xi_j))^T \right]^{-1} \begin{pmatrix} q_P(\xi_j) & 0 \\ 0 & -q_s(\xi_j) \end{pmatrix} (F'_{\Theta}(\xi_j))^{-1}
\]
\[ \hat{M}(\zeta(\xi)) = M(\xi), \zeta(\xi) = \xi^2 \]

and \( \hat{T} = \hat{T}(\xi), \hat{T}(\zeta(\xi)) = T(\xi) \) with

\[ T(\xi) = -\frac{\xi \mu_0}{\pi \omega^2} \left[ (F^a_\Theta)^{-1}(-\xi) \right]^{-1} \begin{pmatrix} q_P(\xi) & 0 \\ 0 & -q_S(\xi) \end{pmatrix} \begin{pmatrix} 0 \\ -q_S(\xi) \end{pmatrix} \]

signifying the branch cut.

\( \alpha_j \) and \( T \) can be expressed in terms of the Jost function only, thus the Lemma indicates that the Jost function encodes the boundary spectral data.
unique recovery

- we assume that $H$, $\lambda_0$, $\mu_0$, $\mu(0)$ and $\mu'(0)$ are known
- introduce the expansion of the Jost solution at the boundary

\[ F(0, \xi) = \xi G_0(0, \xi) + G_1(0) + R(\xi), \quad R(\xi) = O \left( \frac{1}{|\xi|} \right) \]

we can construct explicit expressions for $G_0(0, \xi)$ and $G_1(0, \xi)$

Lemma

*Given $\lambda_0$ and $\mu_0$. The mapping from $G^H$ to $(G_0(0, \xi), G_1(0, \xi))$ for any pair of frequencies, $\omega_1 \neq \omega_2 \in \mathbb{R}_+$, is an injection.*

thus $(G_0(0, \xi), G_1(0, \xi))$ for any pair of frequencies, $\omega_1 \neq \omega_2$ determine $G^H$; moreover, $G^H$ together with $H$, $\lambda_0$, $\mu_0$ and $\omega$ determine $Q_0$
unique recovery

as \( \lambda_0, \mu_0 \) are known, the Jost function determines the Weyl matrix

**Proposition**

Given \( G^H \). For \( \omega \) fixed, let \( V_1, V_2 \) be compactly supported on \([0, H]\) and belong to \( L^1([0, H])\) with associated Weyl matrices \( M_1, M_2 \). If \( H, \lambda_0, \mu_0, \mu(0) \) and \( \mu'(0) \) are known and the Assumptions hold true, then \( M_2(\xi) = M_1(\xi) \) for all \( \xi \in \mathcal{K}_+ \) implies that \( V_2 = V_1 \).

proof: Gel’fand-Levitan-type equation, with some complications

thus, \( G^H \) together with \( M(\xi) \) determine \( V \)
by implication, \((G_0(0, \xi), G_1(0, \xi))\) for any two frequencies \(\omega_1 \neq \omega_2 \in \mathbb{R}_+\) and \(M(\xi)\) determine \(Q\).

**Theorem**

Let \(Q_1, Q_2\) be of Lamé type with associated Jost functions \(F_{\Theta;1}, F_{\Theta;2}\). Assume that \(H, \lambda_0, \mu_0, \mu(0)\) and \(\mu'(0)\) are known. Then \(F_{\Theta;2}(\xi) = F_{\Theta;1}(\xi)\) for all \(\xi \in \mathcal{K}_+\) and any pair of frequencies, \(\omega_1 \neq \omega_2 \in \mathbb{R}_+\), subject to the Assumptions, implies that \(Q_2 = Q_1\).

Furthermore, from a Lamé-type \(Q\) for any pair of frequencies, \(\omega_1 \neq \omega_2 \in \mathbb{R}_+\), one can recover \(\lambda\) and \(\mu\).
unique recovery

by implication, \((G_0(0, \xi), G_1(0, \xi))\) for any two frequencies \(\omega_1 \neq \omega_2 \in \mathbb{R}_+\) and \(M(\xi)\) determine \(Q\)

**Theorem**

Let \(Q_1, Q_2\) be of Lamé type with associated Jost functions \(F_{\theta;1}, F_{\theta;2}\). Assume that \(H, \lambda_0, \mu_0, \mu(0)\) and \(\mu'(0)\) are known. Then \(F_{\theta;2}(\xi) = F_{\theta;1}(\xi)\) for all \(\xi \in \mathcal{K}_+\) and any pair of frequencies, \(\omega_1 \neq \omega_2 \in \mathbb{R}_+\), subject to the Assumptions, implies that \(Q_2 = Q_1\).

furthermore, from a Lamé-type \(Q\) for any pair of frequencies, \(\omega_1 \neq \omega_2 \in \mathbb{R}_+\), one can recover \(\lambda\) and \(\mu\)

- we need both the Weyl matrix and the Jost solution at the boundary for the unique recovery of Lamé parameters
- assuming that \(\lambda_0\) and \(\mu_0\) are known, the Jost function determines the Weyl matrix and the Jost solution at the boundary

reconciling seismology with analysis
(Love and) Rayleigh resonances
leaky modes: Rayleigh resonances

- Rosenbaum (1960)
- Phinney (1961) – theoretical study of leaky waves, referred to as pseudo-$P$ modes
- Haddon (1986) – evaluation of the response of a layered elastic medium to an explosive point source ($\sim$ resolvent) using leaking modes
- Schröder & Scott (2001) – study of complex conjugate roots of the Rayleigh equation
- García-Jerez & Sánchez-Sesma (2014) – $P$-$SV$ leaky waves
- Gao, Xia & Pan (2014)
\[
\frac{d}{dZ} \left( \mu \frac{dw_1}{dZ} \right) + i\xi \left( \frac{d}{dZ} (\mu w_2) + \lambda \frac{dw_2}{dZ} \right) + \left( \omega^2 - \xi^2 (\lambda + 2\mu) \right) w_1 = 0
\]
\[
\frac{d}{dZ} \left( (\lambda + 2\mu) \frac{dw_2}{dZ} \right) + i\xi \left( \frac{d}{dZ} (\lambda w_1 + \mu \frac{dw_1}{dZ}) + (\omega^2 - \xi^2 \mu) w_2 \right) = 0
\]

\( Z \in (-\infty, 0] \), supplemented with the (traction) boundary conditions

\[
\chi_1 = \left( \mu \frac{dw_1}{dZ} + i\xi \mu w_2 \right) \bigg|_{Z=0^-} =: a(w) = 0
\]
\[
\chi_2 = \left( (\lambda + 2\mu) \frac{dw_2}{dZ} + i\xi \lambda w_1 \right) \bigg|_{Z=0^-} =: b(w) = 0
\]

\( Z \) is boundary normal coordinate

notation: use \( \xi \) for both \( |\xi| \in \mathbb{R}_+ \) and its values in \( \mathbb{C} \) following analytic continuation
Lamé parameters normalized by density

Assumption

We let \( \mu \geq \alpha_0 > 0, \ 2\mu + 3\lambda \geq \beta_0 > 0, \lambda, \mu \in C^3(\mathbb{R}^-); \lambda(Z) = \lambda_0, \ \mu(Z) = \mu_0 \) for \( Z \leq -H \).

\( H \) signifies thickness of slab, \( Z_I := -H \)

quasi-momenta

\[
q_P = \sqrt{\frac{\omega^2}{\lambda_0 + 2\mu_0} - \xi^2}
\]

\[
q_S = \sqrt{\frac{\omega^2}{\mu_0} - \xi^2}
\]
Riemann surface basics

- Riemann surface $\mathcal{R}$ is obtained for both $q_P$ and $q_S$ by joining the separate Riemann surfaces for $q_P$ and $q_S$ so that $q_P$ and $q_S$ are single-valued holomorphic functions of $\xi$.
- $\mathcal{R}$ is a four-fold cover of the plane; the sheets of $\mathcal{R}$,

$$\mathcal{R} = \mathcal{R}^{++} \cup \mathcal{R}^{+-} \cup \mathcal{R}^{-+} \cup \mathcal{R}^{--} = \bigcup_{\sigma_1, \sigma_2} \mathcal{R}_{\sigma_1, \sigma_2}, \quad (\sigma_1, \sigma_2) = (\text{sign} \text{Im} q_P, \text{sign} \text{Im} q_S)$$

to a point $\xi \in \mathcal{R}$ we may associate the two values $q_S(\xi), q_P(\xi)$ and can determine a mapping $\mathcal{R} \rightarrow \mathcal{R}$ by its action on $q_S(\xi), q_P(\xi)$; thus, we define mappings, $w_P, w_S$ and $w_{SP} : \mathcal{R} \rightarrow \mathcal{R}$

$$q_S(w_S(\xi)) = -q_S(\xi), \quad q_P(w_S(\xi)) = q_P(\xi)$$
$$q_S(w_P(\xi)) = q_S(\xi), \quad q_P(w_P(\xi)) = -q_P(\xi)$$
$$q_S(w_{SP}(\xi)) = -q_S(\xi), \quad q_P(w_{SP}(\xi)) = -q_P(\xi)$$

relations, between the sheets of the Riemann surface, map a point $\xi \in \mathcal{R}$ to another point in $\mathcal{R}$.
Jost solutions $f_P^\pm$, $f_S^\pm$ for $Z < 0$ satisfy the conditions

$$f_P^\pm = f_P^{\pm,0}, \quad f_S^\pm = f_S^{\pm,0} \quad \text{for} \quad Z < Z_I$$

where

$$f_P^\pm = \begin{pmatrix} f_P^{\pm,1} \\ f_P^{\pm,2} \end{pmatrix} = \begin{pmatrix} \xi \\ \pm q_P \end{pmatrix} e^{\pm i Z q_P}, \quad Z < Z_I$$

$$f_S^\pm = \begin{pmatrix} f_S^{\pm,1} \\ f_S^{\pm,2} \end{pmatrix} = \begin{pmatrix} \pm q_S \\ -\xi \end{pmatrix} e^{\pm i Z q_S}, \quad Z < Z_I$$

extend $\mu(Z)$, $\lambda(Z)$ as even functions to $Z > 0$; with these, extend the system to the real line by abuse of notation, we use the same notation, $f_P^\pm$, $f_S^\pm$, for the Jost solutions satisfying the evenly extended system.
boundary matrix

denote

\[ \mathcal{B} = \begin{pmatrix} a(f_P^-) & a(f_S^-) \\ b(f_P^-) & b(f_S^-) \end{pmatrix}, \quad \mathcal{B} = \mathcal{B}(\xi) \]

signifying the \textit{boundary matrix} representing boundary tractions induced by the Jost solutions.

the boundary matrix determines the Jost function via the inverse Markushevich transform assuming that \( \mu(0), \mu'(0) \) and \( \mu_0 \) are known.

Rayleigh determinant

\[ \Delta = \det \mathcal{B} \]
decomposition into entire functions

define

\[ \vartheta_P = \frac{1}{2} (f_P^+ + f_P^-), \quad \varphi_P = \frac{1}{2q_P} (f_P^+ - f_P^-), \quad \vartheta_S = \frac{1}{2} (f_S^+ + f_S^-), \quad \varphi_S = \frac{1}{2q_S} (f_S^+ - f_S^-) \]

boundary matrix takes the form

\[ \mathcal{B} = \begin{pmatrix} a(\vartheta_P) & a(\vartheta_S) \\ b(\vartheta_P) & b(\vartheta_S) \end{pmatrix} - \begin{pmatrix} a(\varphi_P) & a(\varphi_S) \\ b(\varphi_P) & b(\varphi_S) \end{pmatrix} \begin{pmatrix} q_P & 0 \\ 0 & q_S \end{pmatrix} \]

Rayleigh determinant takes the form

\[ \Delta = d_1 + q_P d_2 + q_S d_3 + q_P q_S d_4 \]

where

\[ d_1 = \det \begin{pmatrix} a(\vartheta_P) & a(\vartheta_S) \\ b(\vartheta_P) & b(\vartheta_S) \end{pmatrix}, \quad d_2 = -\det \begin{pmatrix} a(\varphi_P) & a(\vartheta_S) \\ b(\varphi_P) & b(\vartheta_S) \end{pmatrix} \]

\[ d_3 = -\det \begin{pmatrix} a(\vartheta_P) & a(\varphi_S) \\ b(\vartheta_P) & b(\varphi_S) \end{pmatrix}, \quad d_4 = \det \begin{pmatrix} a(\varphi_P) & a(\varphi_S) \\ b(\varphi_P) & b(\varphi_S) \end{pmatrix}; \]
decomposition into entire functions

\[ \vartheta_P = \frac{1}{2} (f_P^+ + f_P^-), \quad \varphi_P = \frac{1}{2q_P} (f_P^+ - f_P^-), \quad \vartheta_S = \frac{1}{2} (f_S^+ + f_S^-), \quad \varphi_S = \frac{1}{2q_S} (f_S^+ - f_S^-) \]

boundary matrix takes the form

\[ \mathcal{B} = \begin{pmatrix} a(\vartheta_P) & a(\vartheta_S) \\ b(\vartheta_P) & b(\vartheta_S) \end{pmatrix} - \begin{pmatrix} a(\varphi_P) & a(\varphi_S) \\ b(\varphi_P) & b(\varphi_S) \end{pmatrix} \begin{pmatrix} q_P & 0 \\ 0 & q_S \end{pmatrix} \]

Rayleigh determinant takes the form

\[ \Delta = d_1 + q_P d_2 + q_S d_3 + q_P q_S d_4 \]

where

\[ d_1 = \det \begin{pmatrix} a(\vartheta_P) & a(\vartheta_S) \\ b(\vartheta_P) & b(\vartheta_S) \end{pmatrix}, \quad d_2 = - \det \begin{pmatrix} a(\varphi_P) & a(\vartheta_S) \\ b(\vartheta_P) & b(\vartheta_S) \end{pmatrix} \]

\[ d_3 = - \det \begin{pmatrix} a(\vartheta_P) & a(\varphi_S) \\ b(\vartheta_P) & b(\varphi_S) \end{pmatrix}, \quad d_4 = \det \begin{pmatrix} a(\varphi_P) & a(\varphi_S) \\ b(\varphi_P) & b(\varphi_S) \end{pmatrix}; \quad S = \det \begin{pmatrix} a(\varphi_S) & a(\vartheta_S) \\ b(\varphi_S) & b(\vartheta_S) \end{pmatrix} \]
intermediate function, Rayleigh resonances

\[ F(\xi) = \Delta(\xi)\Delta(w_S(\xi))\Delta(w_P(\xi))\Delta(w_{PS}(\xi)). \]

is in a Cartwright class \((\mathbb{C}_{4H})\)

Rayleigh resonance “frequencies” are the zeros of the Rayleigh determinant; they are grouped in sets

\[ \Sigma_{++}, \Sigma_{+-}, \Sigma_{-+}, \Sigma_{--} \]

on the four sheets, \(\mathcal{R}_{++}, \mathcal{R}_{+-}, \mathcal{R}_{-+}, \mathcal{R}_{--}\); that is,

\[
\begin{align*}
\Delta(\xi_j) &= 0, \quad \xi_j \in \Sigma_{++}, \\
\Delta(w_P(\xi_j)) &= 0, \quad \xi_j \in \Sigma_{-+}, \\
\Delta(w_S(\xi_j)) &= 0, \quad \xi_j \in \Sigma_{+-}, \\
\Delta(w_{PS}(\xi_j)) &= 0, \quad \xi_j \in \Sigma_{--}
\end{align*}
\]

the set \(\Sigma_{++}\) corresponds with Regge “bound states”
recovery

- $F$ can be recovered from resonance frequencies (Hadamard factorization)
- $S$ can be recovered from “frequencies” at which no mode conversion occurs

**Conjecture**

The boundary matrix can be recovered from the resonance frequencies and $S$.

recovery follows from applying the theorem for spectral data
Lemma

On the Riemann surface $\mathcal{R}$, the following holds true

\[ f_P^\pm (Z, w_P(\xi)) = f_P^\pm (Z, w_{PS}(\xi)) = f_P^{\mp}(Z, \xi), \]
\[ f_S^\pm (Z, w_S(\xi)) = f_S^\pm (Z, w_{PS}(\xi)) = f_S^{\mp}(Z, \xi). \]
identify $\mathcal{R}^{++}$ where $\text{Im } q_P > 0$, $\text{Im } q_S > 0$ with the physical (or “upper”) sheet for $q_S$

$$\mathcal{K}^{++} = \left\{ \xi \in \mathcal{K}_S = \mathbb{C} \setminus \left( \left[ -\frac{\omega}{\sqrt{\mu_0}}, \frac{\omega}{\sqrt{\mu_0}} \right] \cup \mathbb{i} \mathbb{R} \right) : \text{Re } \xi > 0 \right\}$$

on $\mathcal{K}^{++}$ we have $\text{Im } q_P > \text{Im } q_S$