The geometry of Vaidya spacetimes.

Jean-Philippe Nicolas – joint work with Armand Coudray

1 Introduction

- 1916, K. Schwarzschild finds a solution to the Einstein equation describing the universe around a static and spherically symmetric distribution of mass.
- In 1943, P.C. Vaidya [3] solves a long standing open problem in general relativity: finding a modification of the Schwarzschild metric in order to allow for a radiating mass.
- Vaidya's metric is a spherically symmetric solution to the Einstein equations with matter in the form of null dust.
- Vaidya's spacetime has been used a lot to construct explicit models of gravitational collapse, by gluing it to parts of Minkowski and Schwarzschild's spacetime. See the book by J.B. Griffiths and J. Podolsky [2], section 9.5, for a very clear presentation of the metric and an excellent account of the history of these investigations.
- There has been numerous studies of the geometry of Vaidya's spacetime, but they appear to have been mostly numerical. What I shall present here an analytic study of some geometrical features of Vaidya's spacetime. It is a joint work with Armand Coudray who is also in Brest. The paper has appeared in General Relativity and Gravitation in 2021 [1]. The origin of this work is the observation that Vaidya's spacetime is a simple modification of Schwarzschild's universe for which we know very well the localisation of the event horizons. Can we understand precisely where the horizons are located for Vaidya's spacetime? Then this turned into a general study of some classes of null and timelike curves on Vaidya's spacetime that give a rather precise description of the geometry of this universe.

2 Schwarzschild's spacetime

First non trivial solution of the Einstein equations. Found by Karl Schwarzschild just one year after the publication of general relativity. In Schwarzschild coordinates (t, r, θ, φ)

$$g = \left(1 - \frac{2M}{r}\right) \mathrm{d}t^2 - \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r^2 - r^2 \mathrm{d}\omega^2 \,.$$

Two apparently singular regions : $\{r = 0\}$ and $\{r = 2M\}$. The curvature scalar

$$K = R^{abcd} R_{abcd} = \frac{48M^2}{r^6}$$

blows up at $r = 0$ which is therefore a true curvature singularity.

The set $\{r = 2M\}$ is just a coordinate singularity. We can use more adapted coordinates such as the Eddington-Finkelstein coordinates.

- **The Regge-Wheeler coordinate and optical functions.**

  The function $r_* = r + 2M\log(r-2M)$ is such that

  $$\frac{\mathrm{d}r_*}{\mathrm{d}r} = \left(1 - \frac{2M}{r}\right)^{-1}$$

  and therefore

  $$g = \left(1 - \frac{2M}{r}\right)(\mathrm{d}t^2 - \mathrm{d}r_*^2) - r^2\mathrm{d}\omega^2\,.$$

  This variable is very important for constructing scattering theories on Schwarzschild's spacetime as it is the correct radial coordinate for a comparison with Minkowski space near infinity ; using the variable r would introduce artificial long-range differences. We have two optical functions

  $$u = t - r_*\,,\ v = t + r_*$$

  whose gradients generate future oriented null geodesics (outgoing for u and incoming for v).

  $$g = \left(1 - \frac{2M}{r}\right)\mathrm{d}u\mathrm{d}v - r^2\mathrm{d}\omega^2$$

  we have a double null foliation by spherically symmetric null hypersurfaces generated bu the integral lines of ∇u and ∇v.

- **Outgoing Eddington-Finkelstein coordinates.**

  They are the coordinates (u, r, θ, φ), where $u = t - r_*$

  $$g = \left(1 - \frac{2M}{r}\right)\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}r - r^2\mathrm{d}\omega^2\,.$$

  The metric g does not degenerate at $r = 2M$ however its restriction to $r = 2M$ is a 2-metric instead of a 3-metric, hence $\{r = 2M\}$ is a null hypersurface : the past event horizon $\mathscr{H}^-$.

- **Conformal compactification.**

  Performing an inversion in r, i.e. putting $R = 1/r$ and rescaling the metric as $\hat{g} = R^2 g$, we get

  $$\hat{g} = \left(R^2 - 2MR\right)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}R - \mathrm{d}\omega^2\,.$$

  The hypersurface $\{R = 0\}$ now appears as a regular null hypersurface for $\hat{g}$, similar to the horizon ; it is future null infinity, denoted $\mathscr{I}^+$.

- **Incoming Eddington-Finkelstein coordinates.**

  This construction can be done identically with the coordinates (v, r, θ, φ), where $v = t + r_*$

  $$g = \left(1 - \frac{2M}{r}\right) \mathrm{d}v^2 - 2\mathrm{d}v\mathrm{d}r - r^2 \mathrm{d}\omega^2 \,,$$
  $$\hat{g} = \left(R^2 - 2MR\right) \mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}R - \mathrm{d}\omega^2 \,.$$

  In this manner, we construct $\mathscr{H}^+$ and $\mathscr{I}^-$. Draw a picture.

- **The event horizons are teleological.**

  The future event horizon for instance is defined as the future boundary of the past of $\mathscr{I}^+$. Locating it requires to know the history of the universe up to its infinite future. Conversely, the past event horizon is the past boundary of the future of $\mathscr{I}^-$ and locating it requires to know the history of the universe down to its infinite past.

3 Vaidya's spacetime

3.1 Assumptions, metric, curvature

Vaidya's spacetime can be defined from the expression of the Schwarzschild metric in outgoing or incoming Eddington-Finkelstein coordinates, simply assuming that the mass depends on the retarded (resp. advanced) time. In the outgoing case, we have

$$g = \left(1 - \frac{2M(u)}{r}\right) \mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}r - r^2 \mathrm{d}\omega^2 \,.$$

Of course this no longer satisfies the Einstein vacuum equations, but the remarkable thing is that the error is in the form of the stress energy tensor of null dust radiating along the integral lines of ∇u, i.e. tangent to the lines of constant u and ω. If the energy of the dust is positive, which is a natural assumption, then we would expect the mass to decrease. In the incoming case, positive energy incoming null dust would correspond to an increase in the mass.

The incoming case describes a collapsing black hole or a black hole that is accreting matter. The outgoing case describes an evaporating white hole (this is a classical evaporation). One is simply the time-reversed version of the other.

Note that in our work we assume that

$$M \text{ is a smooth function of the retarded time } u \,. \tag{1}$$

This is not a point of detail, it changes the situation quite seriously. Some people chose for instance to study the complete evaporation of a white hole with a brutal ending, i.e. M reaches zero in finite retarded time and with a non zero derivative. In this case, the mass function is merely Lipschitz (at the brutal end of the evaporation). We have made the choice to avoid such situations. This is based on some physical intuition that may or may not be correct.

We work with the following assumption (draw a picture) :

$$\lim_{u \to \pm\infty} M(u) \to M_\pm \text{ with } 0 \leq M_+ < M_- < +\infty \,. \tag{2}$$

This means that we focus on the case of an evaporating white hole that stabilises in finite or infinite time either to a Schwarzschild black hole of that evaporates completely.

We put

$$F = 1 - \frac{2M(u)}{r} \,.$$

The Weyl tensor has Petrov type D and the double principle null directions are

$$V = \frac{\partial}{\partial r} \,, \; W = \frac{\partial}{\partial u} - \frac{1}{2} F \frac{\partial}{\partial r} \,. \tag{3}$$

These are similar to roots of a polynomial with their multiplicity. They have a natural and easy description in terms of spinors ; the tensorial condition is

$$C_{abc[d}V_{e]}V^b V^c = C_{abc[d}W_{e]}W^b W^c = 0 \,.$$

In vacuum, the Goldberg-Sachs Theorem entails that they generate null geodesics but this does not apply here. We have, as in the Schwarzschild case, two congruences of null lines that describe the skeleton of the conformal curvature. They are in fact geodesics as the principal null vector fields turn out to be the gradients of two optical functions.

- One of these congruences is simply given by the curves of constant u and ω.
- The second one does not have a simple expression in terms of u, r.

The curvature scalar (Kretschmann scalar) is given by

$$k = R_{abcd}R^{abcd} = C_{abcd}C^{abcd} = \frac{48M(u)^2}{r^6} \,. \tag{4}$$

3.2 Where are the event horizons?

This is the question that got us working on this topic in the first place. More precisely, where are the event horizons compared to the Schwarzschild event horizons? Are they beyond or beneath the Schwarzschild horizons, do they coincide on some domains?

First consider a simple case where the evaporation is restricted to a compact interval of retarded time and $M_+ > 0$. Draw a picture. The future horizon is easily described as we are in a Schwarzschild region that encompasses the infinite future of the universe ; it is the $\{r = 2M_+\}$ hypersurface. The past horizon is harder to describe globally. In the past Schwarzschild region, it is described by $r = 2M_-$ but as soon as we enter the radiation region, this stops being valid. However, we know that an event horizon is a null hypersurface. So using the spherical symmetry, we can describe the past event horizon by a simple ODE. It is the hypersurface generated by the family of curves indexed by $\omega \in S^2$:

$$\gamma(u) = (u, r = r(u), \omega) \,, \; u \in \mathbb{R} \,, \tag{5}$$

that are such that

$$r(u) = 2M_- \text{ for } u \leq u_-$$

and have the property of being null, i.e. (Eikonal equation)

$$g(\dot\gamma(u), \dot\gamma(u)) = 1 - \frac{2M(u)}{r(u)} + 2\dot r(u) = 0\,. \tag{6}$$

Hence, the function $r(u)$ satisfies the following ordinary differential equation

$$\dot r(u) = -\frac{1}{2}\left(1 - \frac{2M(u)}{r(u)}\right)\,, \tag{7}$$

with $r > 0$ and $r(u) = 2M_-$ for $u \leq u_-$.

Remark 3.1. *We see from (7) that*

$$\dot\gamma(u) = \left(1, -\frac{1}{2}\left(1 - \frac{2M(u)}{r(u)}\right), 0\right) = \frac{\partial}{\partial u} - \frac{F}{2}\frac{\partial}{\partial r}\,,$$

i.e. the null generator of the past horizon is the principal null vector field W.

If we drop the assumption that the radiation happens only for a finite retarded time interval, we have the following generalised Cauchy problem to solve :

$$\boxed{\text{Find a function } r_h \text{ satisfying (7) and such that } \lim_{u\to-\infty} r_h(u) = 2M_-\,.} \tag{8}$$

Using merely the Cauchy-Lipschitz (or Picard-Lindelöf) Theorem and a priori estimates, we can prove the existence and uniqueness of this solution and get precise results on the location and asymptotic behaviour of r_h and all the other solutions to (7).

3.3 Properties of solutions of (7)

Lemma 3.1. *On an interval $]u_0, u_1[$ on which $\dot M(u)$ does not vanish everywhere, $r(u)$ cannot be identically equal to $2M(u)$.*

and moreover

Lemma 3.2. *Let $(]u_1, u_2[, r)$ be a solution to (7) such that, for a given $u_0 \in]u_1, u_2[$, we have $r(u_0) \geq 2M(u_0)$. Let us assume that $\dot M(u) < 0$ for all $u \in]u_1, u_2[$, then $r(u) > 2M(u)$ on $]u_0, u_2[$.*

For our main theorem, we add the following assumption

$$\dot M(u) < 0 \text{ on }]u_-, u_+[\,,\ \ -\infty \leq u_- < u_+ \leq +\infty\,,\ \dot M \equiv 0 \text{ elsewhere,} \tag{9}$$

in order to avoid the multiplication of cases.

Theorem 1. *Under Assumptions (1), (2) and (9), there exists a unique maximal solution r_h to (7) such that*

$$\lim_{u\to-\infty} r_h(u) = 2M_-\,.$$

- *If either $M_+ > 0$ or $u_+ = +\infty$, r_h exists on the whole real line, $r_h(u) \to 2M_+$ as $u \to +\infty$ and any other maximal solution r to (7) belongs to either of the following two categories:*

  1. *r exists on the whole real line, $r(u) > r_h(u)$ for all $u \in \mathbb{R}$, $\lim_{u\to-\infty} r(u) = +\infty$ and $\lim_{u\to+\infty} r(u) = 2M_+$;*
  2. *r exists on $]u_0, +\infty[$ with $u_0 \in \mathbb{R}$ and satisfies: $r(u) < r_h(u)$ for all $u \in]u_0, +\infty[$, $\lim_{u\to u_0} r(u) = 0$ and $\lim_{u\to+\infty} r(u) = 2M_+$.*

- *If $M_+ = 0$ and $u_+ < +\infty$, r_h exists on an interval $] - \infty, u_0[$ with $u_+ \leq u_0 < +\infty$ and $\lim_{u\to u_0} r_h(u) = 0$. The other maximal solutions are of two types:*

  1. *r exists on $] - \infty, u_1[$ with $u_0 \leq u_1 < +\infty$, $r(u) > r_h(u)$ on $] - \infty, u_0[$, $\lim_{u\to u_1} r(u) = 0$ and $\lim_{u\to-\infty} r(u) = +\infty$;*
  2. *r exists on $]u_1, u_2[$ with $-\infty < u_1 < u_2 \leq u_0$, $r(u) \to 0$ as u tends to either u_1 or u_2 and $r(u) < r_h(u)$ on $]u_1, u_2[$.*

3.4 The two optical functions

The function u is an optical function, which means that its gradient is a null vector field, or equivalently that u satisfies the eikonal equation

$$g(\nabla u, \nabla u) = 0\,. \tag{10}$$

An important property of optical functions is :

Lemma 3.3. *Let ξ be an optical function and denote $\mathcal{L} = \nabla\xi$. The integral curves of $\mathcal{L}$ are geodesics and $\mathcal{L}$ corresponds to a choice of affine parameter, i.e.*

$$\nabla_{\mathcal{L}}\mathcal{L} = 0\,.$$

Proof. The proof is direct :

$$\begin{aligned}
\nabla_{\mathcal{L}}\mathcal{L}^b &= \nabla_{\nabla\xi}\nabla^b\xi\,,\\
&= \nabla_a\xi\nabla^a\nabla^b\xi\,,\\
&= \nabla_a\xi\nabla^b\nabla^a\xi \text{ since the connection is torsion-free,}\\
&= \nabla^b\left(\nabla_a\xi\nabla^a\xi\right) - \left(\nabla^b\nabla_a\xi\right)\nabla^a\xi\,,\\
&= 0 - \nabla_a\xi\nabla^a\nabla^b\xi \text{ since } \nabla\xi \text{ is null and the connection torsion-free,}\\
&= -\nabla_{\nabla\xi}\nabla^b\xi\,. \quad \square
\end{aligned}$$

Proposition 3.1. *The integral lines of V are affinely parametrised null geodesics; they are the outgoing principal null geodesics of Vaidya's spacetime.*

We now establish the existence of a second optical function.

Proposition 3.2. *There exists a function v defined on $\mathbb{R}_u \times]0,+\infty[_r \times S^2_\omega$, depending solely on u and r, such that ∇v is everywhere tangent to the integral lines of W. This means that $g(\nabla v, \nabla v) = 0$, i.e. v is an optical function. The integral lines of W are therefore null geodesics and their congruence generates the level hypersurfaces of v ; they are the incoming principal null geodesics of Vaidya's spacetime.*

Proof. The metric g can be written as

$$g = F\mathrm{d}u\left(\mathrm{d}u + 2F^{-1}\mathrm{d}r\right) - r^2\mathrm{d}\omega^2\,,$$

but contrary to the Schwarzschild case, $\mathrm{d}u + 2F^{-1}\mathrm{d}r$ is not exact. We introduce an auxiliary function $\psi > 0$ and we write

$$g = \frac{F}{\psi}\mathrm{d}u\left(\psi\mathrm{d}u + 2\psi F^{-1}\mathrm{d}r\right) - r^2\mathrm{d}\omega^2\,.$$

Our purpose is to find conditions on ψ that ensure that the 1-form $\alpha := \psi\mathrm{d}u + 2\psi F^{-1}\mathrm{d}r$ is exact. Since we work in the variables (u,r) on the simply connected domain $\mathbb{R}_u \times]0,+\infty[_r$, all that is required is that α be closed, i.e. that

$$\mathrm{d}\alpha = 2\frac{\partial}{\partial u}\left(\frac{\psi}{F}\right) - \frac{\partial \psi}{\partial r} = 0\,.$$

This is equivalent to ψ satisfying an ODE with potential along the integral lines of W which is easy to integrate.

$$\frac{\partial \psi}{\partial u} - \frac{F}{2}\frac{\partial \psi}{\partial r} + \frac{2}{F}\frac{\dot{M}}{r}\psi = 0\,.$$

□

3.5 Case of a complete evaporation in infinite time

We now assume that $M_+ = 0$ and $u_+ = +\infty$. As we have established in Theorem 1, the past event horizon ends up at $r = 0$ as $u \to +\infty$ and so do all the integral curves of (7), i.e. all the incoming principal null geodesics. From this, we infer the following theorem.

Theorem 2. *Whatever the speed at which $M(u) \to 0$ as $u \to +\infty$, we have a null singularity of the conformal structure in the future of our spacetime. More precisely, the Kretschmann scalar k does not remain bounded as $u \to +\infty$ along any integral line of (7).*

Proof. Consider $(]u_0,+\infty[, r)$ a maximal solution to (7), with $u_0 \in \mathbb{R} \cup \{-\infty\}$. Assume that k remains bounded along the integral line as $u \to +\infty$. Then, using (4), so does M/r^3 and it follows that M/r tends to 0 as $u \to +\infty$ along the integral line. This implies in turn that $\dot{r}(u) \to -1/2$ as $u \to +\infty$, which contradicts the fact that $r(u) \to 0$ as $u \to +\infty$. □

Remark 3.2. *If we assume that along the integral lines of (7), $\dot{r}(u)$ has a limit as $u \to +\infty$, this limit is necessarily zero in order not to contradict the fact that $r(u) \to 0$ as $u \to +\infty$. This implies in turn that along the integral line,*

$$\frac{M(u)}{r(u)} \to \frac{1}{2} \text{ as } u \to +\infty\,,$$

i.e.

$$r(u) \simeq 2M(u) \text{ as } u \to +\infty \tag{11}$$

and

$$k \simeq \frac{3}{4M(u)^4} \text{ as } u \to +\infty\,. \tag{12}$$

Question : this singularity is for $u \to +\infty$ but is it really asymptotic or can it be reached in finite proper time? We do not have a complete answer to this question but we have constructed families of uniformly timelike curves that are radial and that reach the singularity in infinite proper time.

4 Conclusion

The analytic information we obtain on the incoming principal null geodesics and the construction of the second optical function are precise enough to allow to study the asymptotic behaviour of fields near null infinity and even in some cases to construct scattering theories (by means of conformal methods). These are projects that Armand is currently developing.

A point of connection with the topic of the conference is the quasi normal modes (i.e. resonances) of dynamic black holes, which appear to be an essentially open problem. Why should we study the resonances on Vaidya's spacetime? Well, black holes are believed to evaporate (this is a quantum phenomenon). If the evaporation is very slow, it is likely that it can be neglected and that the resonances that are activated when some massive body falls into the black hole at those of the stationary spacetime. But if the evaporation is fast enough, maybe the resonances will differ from those of a static black hole spacetime. To understand this, a good model for a first study could be Vaidya's spacetime, although the nature of the evaporation is quite different to the quantum effect. The question is not time symmetric. The best in this case is probably to define resonances on a hyperboloidal foliation that is transverse to $\mathscr{I}^+$. In the case of an infinitely long and complete evaporation, what is the link between the asymptotic singularity and the behaviour of dynamical resonances?

The two cases of a black hole in formation and a white hole that evaporates are both interesting and quite different. Note that one may be the time reflexion of the other, but there is something odd in considering an initial asymptotic singularity from which a black hole emerges... Causality is not time symmetric.

References

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