Boundary conditions for the hypoelliptic Laplacian

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Joint work with S. Shen

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Outline

- The hypoelliptic Laplacian (Bismut)
- Maximal subelliptic estimates (Lebeau)
- Dirichlet and Neumann boundary conditions (N.)
- Boundary conditions for d and commutation with d (Shen, N.)

Bismut's presentation

When (M, g) is a riemannian manifold we may consider the duality between $L^2(M; \Lambda T^*M)$ and $L^2(M; \Lambda TM)$ via

$$\langle t\,,\, s\rangle_{TM,T^*M} = \int_M \overline{t(x)}.s(x)\ dv_g(x)\,.$$

This gives rise to the formal adjoint $\tilde{d}$ of d via

$$\langle \tilde{d}t\,,\, s\rangle_{TM,T^*M} = \langle t\,,\, ds\rangle_{TM,T^*M}\,.$$

If $\phi : TM \to T^*M$ is a (fiberwise) M-isomorphism , extended to $\phi : \Lambda TM \to \Lambda T^*M$ we may define

$$\eta_\phi(U, V) = \overline{U}.(\phi V) \quad , \quad \eta_\phi^*(\omega, \theta) = \overline{\phi^{-1}\omega}.\theta\,,$$

$$\text{and} \qquad \langle s\,,\, s'\rangle_\phi = \int_M \eta_\phi^*(s(x), s'(x))\ dv_g(x)\,.$$

This leads to d^ϕ the formal adjoint of d .
The Hodge codifferential d^* is a particular case when $\phi = g : TM \to T^*M$.
This leads to a generalization of Hodge Laplacian

$$(dd^\phi + d^\phi d) = (d + d^\phi)^2\,.$$

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Bismut's presentation

(Q,g) closed compact riemannian manifold, $\nabla^{Q,g}$ the Levi-Civita connection. $X=T^*Q$

$$T(T^*Q)=\underbrace{T(T^*Q)^H}_{TQ}\oplus\underbrace{T(T^*Q)^V}_{T^*Q}\quad,\quad T^*(T^*Q)=\underbrace{T^*(T^*Q)^H}_{\sim T^*Q}\oplus\underbrace{T^*(T^*Q)^V}_{\sim TQ}\,.$$

$X=T^*Q$ is a symplectic ($\sigma: TX\to T^*X$) and riemannian manifold ($g^{TX}=g\oplus^\perp g^{-1}$).

$$\phi_b=\begin{pmatrix} g & -b\mathrm{Id}\\ b\mathrm{Id} & 0\end{pmatrix}\quad,\quad \phi_b^{-1}=\begin{pmatrix} 0 & b^{-1}\mathrm{Id}\\ -b^{-1}\mathrm{Id} & b^{-2}g\end{pmatrix}\qquad b\in\mathbb{R}^*$$

$$\eta_{\phi_b}(U,V)=g(\pi_X(U),\pi_X(V))+b\sigma(U,V)\,,$$

$$dv_{g^{TX}}=dv_\sigma\overset{loc}{=}|dqdp|\quad x=(q,p)\in X=T^*Q$$

$$\mathfrak{h}(q,p)=\frac{1}{2}g^{ij}(q)p_ip_j\quad,\quad\langle p\rangle_q=\sqrt{1+2\mathfrak{h}}\quad.$$

Bismut's Laplacian equals

$$B_{\mathfrak{h}}^{\phi_b}=\frac{1}{4}\left(d_{\mathfrak{h}}^{\phi_b}+d_{\mathfrak{h}}\right)^2=\frac{1}{4}(d_{\mathfrak{h}}^{\phi_b}d_{\mathfrak{h}}+d_{\mathfrak{h}}d_{\mathfrak{h}}^{\phi_b})$$

$$d_{\mathfrak{h}}=e^{-\mathfrak{h}}de^{\mathfrak{h}}\quad,\quad d_{\mathfrak{h}}^{\phi_b}=e^{\mathfrak{h}}d^{\phi_b}e^{-\mathfrak{h}}\,.$$

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Weitzenbock type formula

REF: Bismut J AMS (2005)

The differential, Bismut's codifferential and Bismut's hypoelliptic Laplacian can be defined for sections of $F = \Lambda T^*X \otimes \pi_X^*\mathfrak{f}$, $\pi_X : X = T^*Q \to Q$.

- $(\mathfrak{f}, \nabla^{\mathfrak{f}}, g^{\mathfrak{f}})$ hermitian bundle with a flat connection.
- (eg. $\mathfrak{f} = \mathbb{C}$, $\nabla^{\mathfrak{f}} = 0$ in $\mathcal{C}^\infty(T^*Q; L(\mathfrak{f}))$, $g^{\mathfrak{f}}(z) = e^{2V(q)}|z|^2$).
- Unitary connection $\nabla^{\mathfrak{f},u} = \nabla^{\mathfrak{f}} + \frac{1}{2}\omega(\nabla^{\mathfrak{f}}, g^{\mathfrak{f}})$.
- $\nabla^{Q,\mathfrak{f}}$ connection on $\Lambda TQ \otimes \Lambda T^*Q \otimes \mathfrak{f}$ made of Levi-Civita and $\nabla^{\mathfrak{f}}$.
- $\nabla = \pi_X^*(\nabla^{Q,\mathfrak{f}})$

$(\underline{e}_i)_{i=1,\dots,d}$ local basis of TQ, $(\underline{e}^j)_{j=1,\dots,d}$ basis of T^*Q,

$$e_i = \pi^*(\underline{e}_i) \in TX^H \quad , \quad \hat{e}^j = \pi_*(\underline{e}^j) \in TX^V$$

$$\text{dual basis} \qquad e^i \in T^*X^H \sim T^*Q \quad , \quad \hat{e}_j \in T^*X^V \sim TQ\,,$$

$$B_{\mathfrak{h}}^{\phi_b} = \frac{1}{4b^2}\left[-\Delta^V + |p|_q^2 - \frac{1}{2}\langle R^{TQ}(e_i, e_j)e_k\,,\, e_\ell\rangle e^i e^j \mathrm{i}_{\hat{e}^k \hat{e}^\ell} + N^V - N^H\right]$$
$$-\frac{1}{2b}\Big[\mathcal{L}_{Y^{\mathfrak{h}}} + \frac{1}{2}\omega(\nabla^{\mathfrak{f}}, g^{\mathfrak{f}})(Y^{\mathfrak{h}}) + \frac{1}{2}e^i \mathrm{i}_{\hat{e}^j}\omega(\nabla^{\mathfrak{f}}, g^{\mathfrak{f}})(e_j)$$
$$+\frac{1}{2}\omega(\nabla^{\mathfrak{f}}, g^{\mathfrak{f}})(e_i)\nabla_{\hat{e}^i}\Big]$$

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REF: Port. Math. (2005) Ann. Inst. Fourier (2007)

Weighted L^2-space: Take $g^{\Lambda T^*X} = \langle p\rangle_q^{-N_H+N_V}\pi_X^*(g^{\Lambda T^*Q}\otimes g^{\Lambda TQ})$

$$e^i = \underbrace{dq^i}_{\propto\langle p\rangle_q^{-1/2}} \quad , \quad \hat{e}_j = \underbrace{dp_j}_{\propto\langle p\rangle_q^{1/2}} - \Gamma_{ji}^k(q)p_k \underbrace{dq^i}_{\propto\langle p\rangle_q^{-1/2}}$$

and $dv_{g^{TX}} = |dqdp|$.

Order of differential operators: $\frac{\partial}{\partial q^i} \ : \ 1 \quad , \quad \frac{\partial}{\partial p_j} \ : \ \frac{1}{2} \quad , \quad p_j\times \ : \ \frac{1}{2}$.

Lebeau's presentation

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$$e_i = \underbrace{\frac{\partial}{\partial q^i}}_{\text{order }1} + \Gamma_{ij}^k(q)\underbrace{p_k}_{\text{order }\frac{1}{2}}\underbrace{\frac{\partial}{\partial p_j}}_{\text{order }\frac{1}{2}} \quad , \quad \hat{e}_j = \underbrace{\frac{\partial}{\partial p_j}}_{\text{order }\frac{1}{2}} \quad , \quad \underbrace{\langle p\rangle\frac{\partial}{\partial p_j}}_{order 1}$$

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Sobolev spaces : $\mathcal{W}^r(X;F)$:

$$\cap_r \mathcal{W}^r(X;F) = S(X;F) \quad , \quad \cup_r \mathcal{W}^r(X;F) = S'(X;F) .$$

$$(u\in\mathcal{W}^n(X;F)) \Leftrightarrow \left(\langle p\rangle_q^{2n_1}(\partial_q)^\alpha(\langle p\rangle\partial_p)^\beta u \in L^2(X;F) , |\alpha|+|\beta|+n_1\leq n\right) .$$

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Symbols: $M(q,p)$ symbol of order m iff

$$\|\partial_q^\alpha\partial_p^\beta M(q,p)\|_{L(F)} \leq C_{\alpha,\beta}\langle p\rangle_q^{m-|\beta|} \,,$$

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Geometric Kramers-Fokker-Planck operator:

$$-b\nabla_{Y_{\mathfrak{h}}} + \mathcal{O} + \mathcal{M}$$

$$Y_{\mathfrak{h}} = g^{ij}(q)p_je_i = g^{ij}(q)p_i\left(\frac{\partial}{\partial q^i} + \Gamma_{i\ell}^k(q)p_k\frac{\partial}{\partial p_k}\right),$$

$$\mathcal{O} = \frac{-\Delta_V + |p|_q^2}{2} = \frac{-g_{ij}(q)\partial^2_{p_ip_j} + g^{ij}(q)p_ip_j}{2}$$

$$\mathcal{M} = \mathcal{M}_{0,j}\nabla_{\frac{\partial}{\partial p_j}} + \mathcal{M}_{0,i}p_i + \mathcal{M}_{0,0}\,,\quad \mathcal{M}_{0,*} \text{ symbols of order } 0\,.$$

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Geometric Kramers-Fokker-Planck operator:

$$\underbrace{-b\nabla_{Y_{\mathfrak{h}}}}_{\text{order } \frac{3}{2}} + \underbrace{\mathcal{O}}_{\text{order } 1} + \underbrace{\mathcal{M}}_{\text{order } \frac{1}{2}}$$

$$Y_{\mathfrak{h}} = g^{ij}(q)p_je_i = g^{ij}(q)p_i\Big(\frac{\partial}{\partial q^i} + \Gamma_{i\ell}^k(q)p_k\frac{\partial}{\partial p_k}\Big)\,,$$

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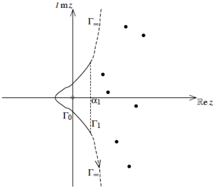
Maximal subelliptic estimate

REF: *Max. Hypo.* Lebeau Ann. Inst. Fourier (2007), Helffer-Nourrigat (1985), Hörmander (Book IV-Chap 27)
"cuspidal" semigroup: Bismut-Lebeau (2008), Hérau-N. (2004), Helffer-N. (2005), N. (2018)

By using the metric $g^{\Lambda T^*X} = \langle p\rangle_q^{-N_H+N_V}\pi_X^*(g^{T^*Q}\otimes g^{\Lambda TQ})$, Bismut's Laplacian $2b^2 B_{\mathfrak{h}}^{\phi_b}$ ($b \in \mathbb{R}^*$ fixed) is a GKFP-operator.

The operator $K = C_b + 2b^2 B_{\mathfrak{h}}^{\phi_b}$ is cuspidal $e^{-tK} = \frac{1}{2i\pi}\int_\Gamma \frac{e^{-tz}}{(z-K)}\,dz$ for $t > 0$.



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There exists $C_b > 0$ and for any $r \in \mathbb{R}$, $C_{r,b} > 0$ such that

$$\|\mathcal{O}s\|_{\mathcal{W}^r} + \|\nabla_{Y^{\mathfrak{h}}} s\|_{\mathcal{W}^r} + \|s\|_{\mathcal{W}^{r+2/3}} + \langle\lambda\rangle^{1/2}\|s\|_{\mathcal{W}^r} \leq C_{r,b}\|(C_b + 2b^2 B_{\mathfrak{h}}^{\phi_b} - i\lambda)s\|_{\mathcal{W}^r} .$$

The operator $C_b + 2b^2 B_{\mathfrak{h}}^{\phi_b}$ is maximal accretive endowed with

$$D(2b^2 B_{\mathfrak{h}}^{\phi_b}) = \left\{ s \in L^2(X; F) , \quad B_{\mathfrak{h}}^{\phi_b} s \in L^2(X; F) \right\} .$$

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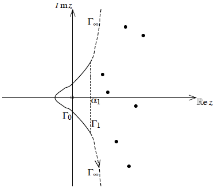

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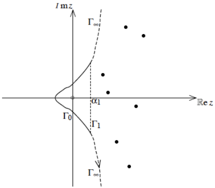

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$$g^{TQ} = (dq^1)^2 + m(|q^1|, q') = (dq^1)^2 + m_{i'j'}(|q^1|, q')dq^{i'}dq^{j'}\,, 1 \notin \{i', j'\}\,.$$

The flat case corresponds to $m = m(0, q')$ (totally geodesic boundary).

For the elliptic Laplacian on $\overline{Q}_-$, Dirichlet boundary conditions correspond to odd elements of $D(\Delta_Q^{Hodge})$ for the involution $(q^1, q') \to (-q^1, q')$ and Neumann boundary condition to even elements of $D(\Delta_Q^{Hodge})$.

Note that in the general case $\partial_{q^1} m$ is not continuous and the Christoffel symbols Γ_{ij}^k are discontinuous along $Q' = \left\{\underline{q}^1 = 0\right\}$

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$$\Sigma_*(s^J_I(q^1, q', p_1, p')e^I\hat{e}_J) = (-1)^{|\{1\}\cap I|+|\{1\}\cap J|}s^J_I(-q^1, q', -p_1, p')e^I\hat{e}_J .$$

Proposal of boundary conditions for $B^{\phi_b}_{\mathfrak{h}}$ on $\overline{X}_- = \pi_X^{-1}\overline{Q}_-$, in the general case

$$s = s^J_I(q^1, q', p_1, p')e^I\hat{e}_J$$

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But remember $\hat{e}_j = dp_j - \Gamma_{ij}^k(q)p_k dq^i$ are not continuous in the general case .

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$$\left(s\in D(\overline{B}^{\phi_b}_{\mathfrak{h},\mp})\right)\Leftrightarrow\left(s_{ev}\in D(B^{\phi_b}_{\mathfrak{h}})\right)$$
$$s_{ev}=s1_{X_-}(x)\mp(\Sigma_* s)1_{X_+}(x)\,.$$

In the flat case Q is a $\mathcal{C}^\infty$ closed and compact riemannian manifold. $B^{\phi_b}_{\mathfrak{h}}$ is the usual Bismut's Laplacian. Lebeau's maximal subelliptic estimates ensure $D(B^{\phi_b}_{\mathfrak{h}})\subset\mathcal{W}^{2/3}(X)$.

With $2/3>1/2$, any $s\in D(B^{\phi_b}_{\mathfrak{h}})$ (in particular s_{ev}) has a trace along $X'=\pi_X^{-1}(Q')$.

Additionally for all $s\in D(\overline{B}^{\phi_b}_{\mathfrak{h},\mp})$,

$$\|\mathcal{O}s\|+\|\mathcal{Y}s\|+\|s\|_{\mathcal{W}^{2/3}}+\langle\lambda\rangle^{1/2}\|s\|+\|s\big|_{X'}\|_{L^2(X')}\leq C_b\|(C_b+2b^2\overline{B}^{\phi_b}_{\mathfrak{h},\mp}-i\lambda)s\|\,.$$

where the measure on X' is $|p_1||dq'dp|$.

Subelliptic estimate with a totally geodesic boundary

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Subelliptic estimate general boundary

REF: N. Mem. AMS (2018)

Keep the boundary conditions

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One defines a closed maximal accretive operator $\overline{B}^{\phi_b}_{\mathfrak{h},\mp}$ in $L^2(X_-,F)$ with those boundary conditions. Moreover the following estimate holds for all $s \in D(\overline{B}^{\phi_b}_{\phi_b,\mp})$.

To be compared with the flat case (or Lebeau's result for closed compact manifold)

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The curvature problem

When $\partial_{q^1} m(0^-, q') \neq 0$ the Christoffel symbols and actually the second fundamental form are discontinuous for the continuous metric $(dq^1)^2 + m(|q^1|, q')$.

At first sight the frames

$$e^i_{\mp} = dq^i \quad , \quad \hat{e}_{j,\mp} = dp_j - \Gamma^k_{j,i}(0^{\mp}, q')p_k dq^i$$

are discontinuous.

This is solved by using $g\big|_{\partial Q'} = g_0\big|_{\partial Q'}$, $g^{TX} = \pi_X^*(g^{TQ} \oplus g^{T^*Q})$ and identifying $\hat{e}_{j,-}$ with $\hat{e}_{j,+}$ along X'. Parallel transport along e_1 on both sides allows to introduce a continuous piecewise $\mathcal{C}^\infty$ vector bundle structure for which traces of smooth enough elements makes sense.

This is used in two steps,

- Parallel transport on $X = T^*Q$ provides non symplectic coordinates $(\tilde{q}, \tilde{p})$ such that $g^{ij}(q)p_ip_j = g_0^{ij}(\tilde{q})\tilde{p}_i\tilde{p}_j$.
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Lebeau's spaces $\mathcal{W}^s$ are preserved for $s \in [-1, 1]$ by those changes of gauge.

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The curvature problem

However one ends with GKFP operator with discontinuous coefficients in the perturbative term

$$\mathcal{M} = \mathcal{M}_{2,j}\nabla_{\partial_{p_j}} + \mathcal{M}_{2,0}$$

where the $\mathcal{M}_{2,*}$ are more over symbols of degree 2 (on both sides $\overline{X}_-$ and $\overline{X}_+$) .

The vertical weight is treated first and one can prove via an integration by part and conjugation with $\langle p\rangle_q^n$,

$$\langle p\rangle_q^{n+1}(C_b + 2b^2\overline{B}^{\phi_b}_{g,\mathfrak{h}})^{-1}\langle p\rangle_q^{-n} \in \mathcal{L}(L^2(X;F))\,.$$

Lebeau's maximal subelliptic estimate with the exponent $2/3 > 1/2$ is now crucial while using some bootstrap regularity arguments after applying several resolvents, with

$$\|\mathcal{M}_{0,j}\nabla_{\partial_{p_j}} s\|_{\mathcal{W}^{r-1/2}} \leq \|s\|_{\mathcal{W}^r} \quad r - 1/2 \geq 1/6 (r \geq 2/3)\,,$$

combined with the one dimensional multiplication rule for Sobolev spaces

$$\varphi \in W^{r-1/2,2}(\mathbb{R}) \Rightarrow (1_{\mathbb{R}_+}(q^1)\varphi \in W^{r-1/2-0,2}(\mathbb{R})))\,, r - 1/2 \geq 1/6 > 0\,.$$

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$$\varphi \in W^{r-1/2,2}(\mathbb{R}) \Rightarrow (1_{\mathbb{R}_+}(q^1)\varphi \in W^{r-1/2-0,2}(\mathbb{R}))\,, r - 1/2 \geq 1/6 > 0\,.$$

The curvature problem

However one ends with GKFP operator with discontinuous coefficients in the perturbative term
$$\mathcal{M} = \mathcal{M}_{2,j}\nabla_{\partial_{p_j}} + \mathcal{M}_{2,0}$$
where the $\mathcal{M}_{2,*}$ are more over symbols of degree 2 (on both sides $\overline{X}_-$ and $\overline{X}_+$) .

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Boundary conditions for $\overline{d}_{g,\mathfrak{h}}$ and $d^{\phi_b}_{g,\mathfrak{h}}$

REF: Joint work with S. Shen (21)

When $s \in L^2_{loc}$ and $ds \in L^2_{loc}$, s admits partial (tangential) traces along any hypersurface.

The boundary condition that we take for the differential are

$$s^J_{I'}(0,q',p_1,p') = \mp(-1)^{|J\cap\{1\}|} s^J_{I'}(0,q',-p_1,p') \quad 1 \notin I'$$

and for Bismut's codifferential

$$s^{\{1\}\cup J'}_I(0,q',p_1,p') = \mp(-1)\times(-1)^{|\{1\}\cap I|} s^{\{1\}\cup J'}_I(0,q',-p_1,p')\,.$$

Those boundary conditions lead to closed realization of $\overline{d}_{g,\mathfrak{h},\mp}$ and $\overline{d}^{\phi_b}_{g,\mathfrak{h},\mp}$ (adjoint to each other for ϕ^b or ${}^t\phi^b$ duality products).

Unusual thing: The boundary conditions on $\overline{d}_{g,\mathfrak{h},\mp}$ depend on the metric g^{TQ}.

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True: $\overline{d}_{g,\mathfrak{h}} \circ \overline{d}_{g,\mathfrak{h}} = 0$, $\overline{d}^{\phi_b}_{g,\mathfrak{h}} \circ \overline{d}^{\phi_b}_{g,\mathfrak{h}} = 0$.

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Commutation with the resolvent

REF: Joint work with S. Shen (21),
Amrein-Boutet de Monvel-Georgescu (C^0-group and commutator techniques)

For any $t > 0$, $e^{-t\overline{B}^{\phi_b}_{\mathfrak{h},\mp}}$ sends $L^2(X;F)$ to $D(\overline{d}_{\mathfrak{h},\mp}) \cap D(\overline{d}^{\phi_b}_{g,\mathfrak{h},\mp})$ and

$$\forall s \in D(\overline{d}_{g,\mathfrak{h},\mp}), \quad e^{-t\overline{B}^{\phi_b}_{\mathfrak{h},\mp}}\overline{d}_{g,\mathfrak{h},\mp}s = \overline{d}_{g,\mathfrak{h},\mp}e^{-t\overline{B}^{\phi_b}_{\mathfrak{h},\mp}}s$$

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Consequence: For any $z \notin \mathrm{Spec}(\overline{B}^{\phi_b}_{g,\mathfrak{h}})$,

$$\forall s \in D(\overline{d}_{g,\mathfrak{h},\mp}) \quad (z - \overline{B}^{\phi_b}_{g,\mathfrak{h}})^{-1}\overline{d}_{g,\mathfrak{h}}s = \overline{d}_{g,\mathfrak{h}}(z - \overline{B}^{\phi_b}_{g,\mathfrak{h}})^{-1}s\,,$$

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