# Quantum resonances in presence of classical chaos

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(joint works with M.Zworski & J.Sjöstrand)

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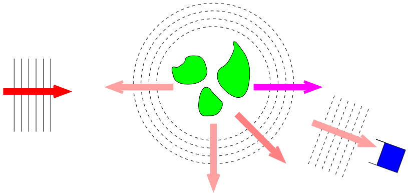
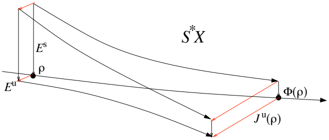
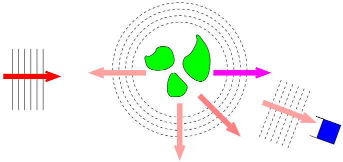


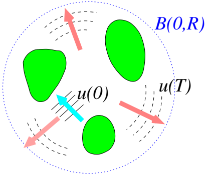


# Outline

- Quantum / scalar wave on $\mathbb{R}^d$, scattered by obstacles or nonflat metrics $\rightsquigarrow$ resonance spectrum.
- High frequency (= semiclassical) regime $\rightsquigarrow$ relevance of the classical (ray) dynamics: geodesic flow.
  Distribution of resonances at high frequency $\rightsquigarrow$ set of *trapped geodesics*.
- Focus on *hyperbolic* dynamics: assume the trapped orbits are *exponenially unstable* $\rightsquigarrow$ they form a *fractal* set, carrying a *chaotic flow*.
  - hyperbolicity $\rightsquigarrow$ *fast dispersion* of the waves; on the other hand, constructive interferences could keep them localized.
    Dynamical conditions for a *resonance gap*? (=global upper bound on resonance lifetimes)
  - Counting high frequency resonances : fractal Weyl's law?

# Wave scattering and decay






Scalar waves scattered by obstacles $\mathcal{O} \Subset \mathbb{R}^d$:

$$(\partial_t^2 - \Delta_\Omega)u = 0, \quad u(0) = u_0,\ \partial_t(0) = u_1,$$

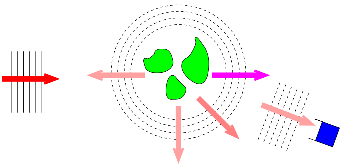
$\Delta_\Omega$ Dirichlet Laplacian on $\Omega = \mathbb{R}^d \setminus \mathcal{O}$ ($\Omega$ connected, $\partial\Omega$ smooth).

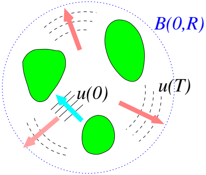
Compactly supported initial data $u_0,\ u_1 \in C_c^\infty(\Omega)$: both

- the local energy $\mathcal{E}_R(u(t)) = \frac{1}{2}\int_{B(0,R)}(|\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2)\,dx$
- the *"correlation"* $\langle f, u(t)\rangle_{\mathcal{D},\mathcal{D}'}$, for a given $f \in C_c^\infty(\Omega)$,

will decay to zero as $t \to \infty$.

# Wave scattering and decay



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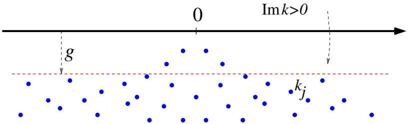
will decay to zero as $t \to \infty$.

*Can we better describe this decay? How does it depend on the obstacles?*

- Large time asymptotics $\rightsquigarrow$ spectral problem.

# Resolvent and Resonances (odd dimension)




Central object in spectral theory: the *resolvent* $R_\Omega(\lambda) = (-\Delta_\Omega - \lambda^2)^{-1}$.

• $R_\Omega(\lambda)$ can be defined in $\{\mathrm{Im}\,\lambda > 0\}$ ("physical sheet"), blows up when $\mathrm{Im}\,\lambda \searrow 0$ (cf. continuous spectrum).
Yet, the truncated resolvent $\chi R_\Omega(\lambda)\chi : L^2 \to L^2$ can be *meromorphically continued* into $\{\mathrm{Im}\,\lambda < 0\}$ ("unphysical sheet").
$\rightsquigarrow$ Resonances $\{\lambda_j\}$ = discrete poles of finite multiplicities.

• If the strip $\{0 \geq \mathrm{Im}\,\lambda > -A\}$ contains finitely many resonances, one hopes to show expansions of the type (with $u(0) = 0, \partial_t u(0) = u_1$) as:

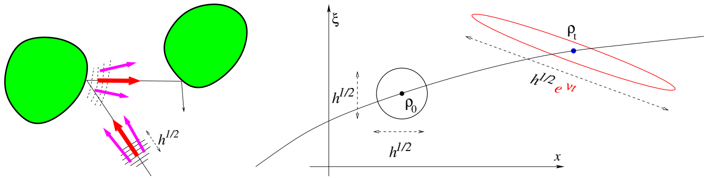
$$\langle f, u(t)\rangle = \sum_{\mathrm{Im}\,\lambda_j > -A} e^{-it\lambda_j} \langle f, v_j\rangle\langle v_j, u_1\rangle + \mathcal{O}(e^{-tA}), \quad t \to \infty.$$

$v_j \in C^\infty(\Omega)$ the resonant state associated with $\lambda_j$, of lifetime $|\mathrm{Im}\,\lambda_j|^{-1}$.
Note that $v_j \notin L^2(\Omega)$ (diverges exponentially when $|x| \to \infty$).

# High frequency / semiclassical regime

- Half-wave equation $i\partial_t u = \sqrt{-\Delta_\Omega} u \equiv$ semiclassical Schrödinger equation $ih\partial_t u = P_h u$, with quantum Hamiltonian $P_h = h\sqrt{-\Delta_\Omega}$.
High-$\lambda$ regime $\equiv$ semiclassical regime $h \sim \lambda^{-1} \ll 1$ at energies $\sim 1$.

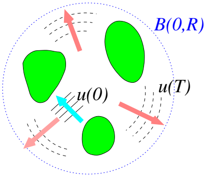
$\Longrightarrow$ **Semiclassical analysis**: the quantum dynamics is guided by the *classical dynamics* on the *phase space* $\Omega \times \mathbb{R} = \{\rho = (x, \xi)\}$ generated by the Hamiltonian $p(x, \xi) = |\xi|$: the geodesic flow $\Phi^t$ on $\Omega \times \mathbb{R}$.

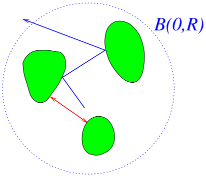



Ex: wavepacket $u_{\rho_0}(x) = a(\frac{x-x_0}{\sqrt{h}})\, e^{i\frac{\xi_0 \cdot x}{h}}$ localized near $x_0$; its $h$-Fourier transform is localized near $\xi_0$.
$\equiv u_{\rho_0}$ is microlocalized in the $\sqrt{h}$-nbhd of $\rho_0 = (x_0, \xi_0)$.

$\Longrightarrow$ semiclassical correspondence: $u(t) = e^{-itP_h/h} u_{\rho_0}$ is a wavepacket microlocalized near $\rho_t = \Phi^t(\rho_0)$.
microscopic shape deformed by $d\Phi^t(\rho_0)$: dispersion.

# Distribution of resonances vs. classical trapped set



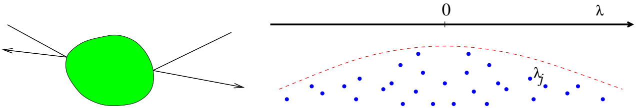

Classical mechanical problem: understand the long time behaviour of the *ray dynamics* on the energy shell $p^{-1}(1) = S^*\Omega$.

Focus on the interaction region $B(0, R)$:

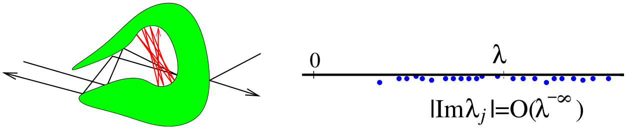
- most trajectories spend a finite time in the interaction region before escaping to $|x| \to \infty$
- there may exist trapped trajectories: **trapped set** $K \stackrel{\text{def}}{=} \{\rho \in S^*\Omega,\ \Phi^t(\rho) \not\to \infty \text{ when } t \to \pm\infty\}$
  Compact, flow-invariant subset of $S^*\Omega$.

**Main idea**: the distribution of resonances near the real axis depends on the geometric and dynamical properties of $K$.

# Distribution of resonances vs. trapped set (2)

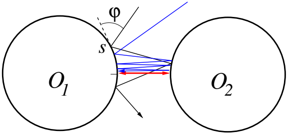


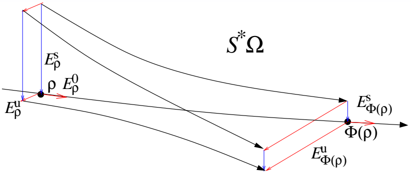

**Case 0:** $K$ empty (ex: convex obstacle).
**No** resonances in $\{|\operatorname{Im}\lambda| \leq C \log \operatorname{Re}\lambda\}$.
[Lax-Phillips, Vainberg, Morawetz, Melrose, Ralston, Strauss]




**Case 1:** $K$ contains an elliptic (=stable) periodic orbit.
Low dispersion $\Longrightarrow$ one can construct quasimodes
$\|(\Delta_\Omega + \lambda^2) v_\lambda\| = \mathcal{O}(\lambda^{-\infty})$ microlocalized on $K$, and identify nearby resonances with $|\operatorname{Im}\lambda_j| = \mathcal{O}(\lambda^{-\infty})$
[Ralston, Lazutkin, Popov, Vodev, Stefanov, Tang-Zworski]

# $K$ a single hyperbolic orbit



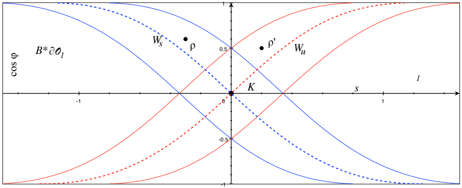




Case 2: 2 convex obstacles (on $\mathbb{R}^2$). $K$ = single hyperbolic periodic orbit $\gamma$.

**Hyperbolicity:** stable $(E^s)$ / unstable $E^u$ directions transverse to the orbit, tangent to the stable $(W^s)$ and unstable $(W^u)$ manifolds.

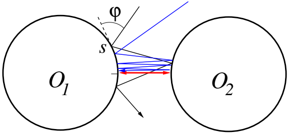
$$\forall t > 0, \qquad \|d\Phi^t \restriction_{E^s_\rho}\| \leq C\, e^{-\nu t}, \qquad \|d\Phi^{-t} \restriction_{E^u_\rho}\| \leq C\, e^{-\nu t}.$$

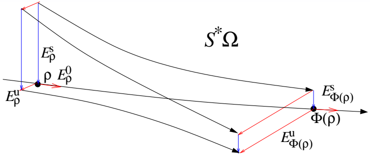
Quantitatively: $d\Phi^t \restriction_{E^u_\rho} \sim e^{t\nu_\gamma}$, with $\nu_\gamma > 0$ the Lyapunov exponent.




Visualization of $W^s/W^u$ on the *Poincaré dynamics* on $\partial O_1$. The stripes correspond to points bouncing $1 \to 2$ in the future or the past.

# $K$ a single hyperbolic orbit (2)



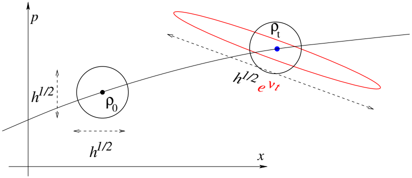



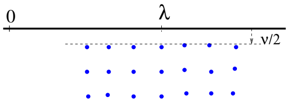

2 convex obstacles (on $\mathbb{R}^2$).

[IKAWA, GÉRARD, GÉRARD-SJÖSTRAND, SJÖSTRAND]

- Hyperbolicity of $\gamma \Longrightarrow$ fast dispersion (=deformation) of a wavepacket on $\gamma$:

$$|\langle u_\rho, e^{-itP_h/h} u_{\rho_\gamma} \rangle| \leq C\, e^{-t\nu_\gamma/2}, \qquad \forall t > 0$$



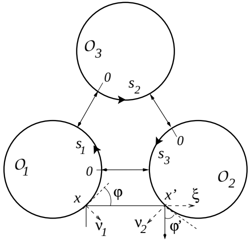



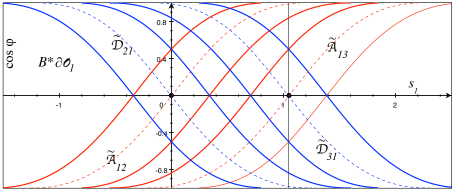

$\rightsquigarrow$ high-frequency gap $\nu_\gamma/2$ of the resonance spectrum.

A more precise analysis (Quantum Birkhoff normal form) provide asymptotic values for the resonances near $\lambda \gg 1$: they form a deformed half-lattice.

# Chaotic trapped set

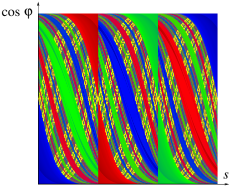
Case 3: $N \geq 3$ convex obstacles in $\mathbb{R}^d$ (non-eclipse condition)

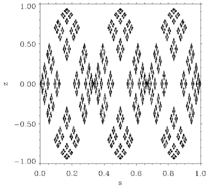



- Every orbit of $K$ is **hyperbolic**, splitting $E^u_\rho \oplus E^s_\rho \oplus V(\rho)$.
  Unstable Jacobian $J^u_t(\rho) = |\det(d\Phi^t \restriction_{E^u_\rho})| \sim e^{\Lambda_\rho t}$
- **Complexity**: $K$ contains infinitely many orbits. $K$ a **fractal repeller**

Each colored strip: points escaping the system at time $0, 1, 2, \ldots$. After removing those strips, remains the trapped set (right).

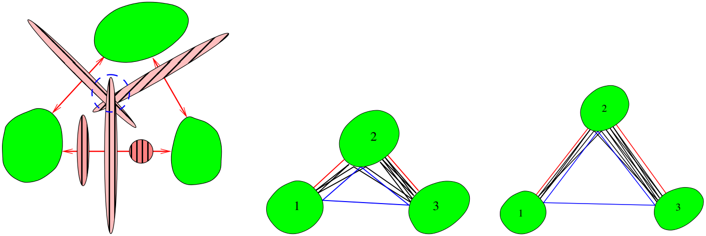



(Plot ©Leon Poon)

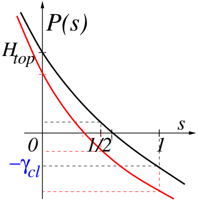
# Chaos: hyperbolicity + complexity

An initial wavepacket $u_\rho$ sitting on a trapped point $\rho \in K$ will spread along $W^u$. Some parts will escape, some will land on other trapped orbits. **Interferences** between different wavepackets may slow down the escape




This competition between hyperbolicity and complexity can be measured by topological pressures:

$$\mathcal{P}(s) \stackrel{\text{def}}{=} \lim_{T\to\infty} \frac{1}{T} \log \sum_{\gamma: T \leq T_\gamma \leq T+1} (J_\gamma^u)^{-s}$$

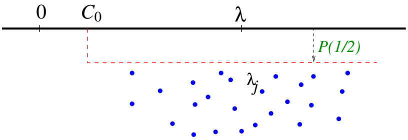


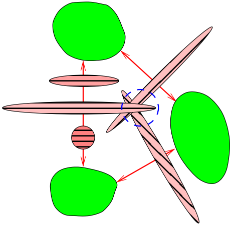

The *thinner* $K$, the smaller $\mathcal{P}(s)$. We will be interested in $\mathcal{P}(s = 1/2)$.

# Resonance gap for thin chaotic trapped sets

## Theorem ([Ikawa, Gaspard-Rice, Burq, N-Zworski])

*Assume the trapped set $K$ is a hyperbolic repeller with $\mathcal{P}(1/2) < 0$. Then $\forall \epsilon > 0$, the strip $\{0 \geq \operatorname{Im} \lambda \geq \mathcal{P}(1/2) + \epsilon\}$ contains at most finitely many resonances. Resonance gap.*

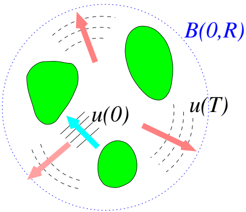


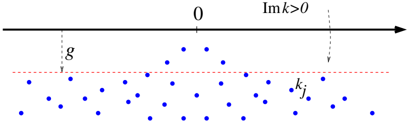



Proof: show that any high-frequency initial state $u_\lambda \in C_c^\infty(B(0,R))$ decays at least like $C\, e^{t\mathcal{P}(1/2)}$:

1. hyperbolicity $\Longrightarrow$ wavepackets microlocalized on $K$ disperse at the rate $(J_t^u)^{-1/2}$, thus "leak out" of $B(0,R)$ after a time $C_1 \log \lambda$.
2. Sum the contributions of many trajectories. If $\mathcal{P}(1/2) < 0$, dispersion always beats interferences $\Longrightarrow$ leakage at rate $\|u_\lambda(t)\|_{L^2(B(0,R))} \leq C e^{(t - C_1 \log \lambda)\mathcal{P}(1/2)}$. □

# Dynamical consequences of the resonance gap ($d \geq 3$ odd)



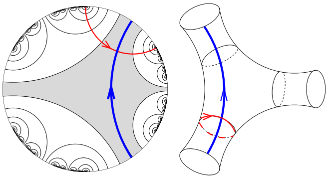

- Exponential decay of the local energy [MORAWETZ-RALSTON-STRAUSS, IKAWA, CHRISTIANSON].
  $\forall s > 0$, $\exists \alpha_s, C > 0$, *for any* $u_0 \in H^s(\Omega)$ *supported in* $B(0,R)$,

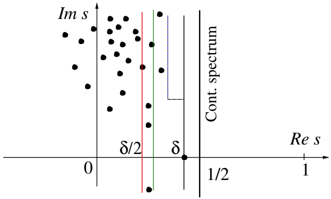
$$\mathcal{E}_R(u(t)) \leq C e^{-\alpha_s t} \|u_1\|^2_{H^s}, \quad \forall t > 0 \,.$$

- For any $f \in C_c^\infty(\Omega)$, the correlation function

$$\langle f, u(t) \rangle = \sum_{\mathrm{Im}\,\lambda_j > -A} e^{-it\lambda_j} \langle f, v_j \rangle \langle v_j, u_1 \rangle_{\mathcal{D}', \mathcal{D}} + \mathcal{O}_f(e^{-tA}) \|u_1\|_{H^N} \,.$$

# How sharp is the pressure bound?






Geometric models of scattering : $X = \Gamma\backslash\mathbb{H}^2$ surface of constant negative curvature and infinite area ($\to$ hyperbolic flow).
Resonances $\lambda_j^2 = s_j(1-s_j)$, with $s_j$ zero of the Selberg zeta function.

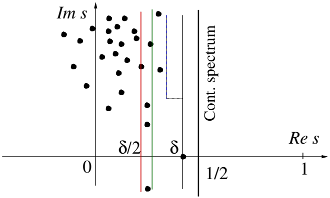
- $\mathcal{P}(1/2) < 0 \longleftrightarrow \delta < 1/2$, where $\delta = \frac{\dim K - 1}{2}$.

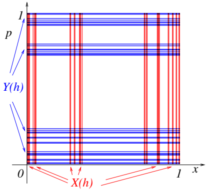
Pressure bound: $s_j < \delta$ [PATTERSON,SULLIVAN], nontrivial for $\delta < 1/2$.

Improvements on this pressure bound:

- [NAUD] uses *Dolgopyat's method* to prove partial cancellations when summing over trajectories $\Longrightarrow \mathrm{Re}\, s_j \leq \delta - \epsilon_1$.
- [PETKOV-STOYANOV] apply the same method to $N$-obstacle scattering on $\mathbb{R}^d$ (non-eclipse condition) $\Longrightarrow \mathrm{Im}\, \lambda_j \leq \mathcal{P}(1/2) - \epsilon_1$.

# Resonance gap for "thick" chaotic trapped sets




*Fractal Uncertainty Principle* [BOURGAIN-DYATLOV].
$X$, $Y \subset [0,1]$ **fractal** sets, $X(h)$, $Y(h)$ their $h$-neighbourhoods.
Then $\exists \beta = \beta(X,Y) > 0$ s.t., for $0 < h \ll 1$,

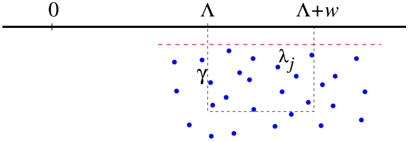
$$\| \mathbb{1}_{X(h)}(x)\, \mathbb{1}_{Y(h)}(-ih\nabla_x) \|_{L^2 \to L^2} \leq h^{\beta}.$$

Replacing horizontal / vertical leaves by $W^u/W^s$ by , and using the fractal structure of $K$, one obtains resonance gaps without conditions on $\mathcal{P}(1/2)$:

- [DYATLOV-ZAHL,BOURGAIN-DYATLOV,JIN-ZHANG] Resonance gap on $\Gamma\backslash\mathbb{H}^2$ for any value of $\delta \in (0,1)$:
  $\exists \epsilon(\delta) > 0$, $\mathrm{Re}\, s_j \leq \frac{1}{2} - \epsilon(\delta)$ for $\mathrm{Im}\, s_j \geq C$.
- [VACOSSIN (WIP)] $\exists$ resonance gap for $N$-obstacle scattering on $\mathbb{R}^2$, whatever the thickness of the hyperbolic repeller $K$.

# Counting long-living resonances for a chaotic $K$

Below the gap, how many resonances are there?




**Theorem** ([SJÖSTRAND'90, SJÖSTRAND-ZWORSKI'07, N-SJ-ZW'11])

*The number resonances in a high-frequency box is bounded above by:*

$$\forall \lambda \geq \Lambda_0, \quad \# \{\lambda \leq \operatorname{Re} \lambda_j \leq \lambda + w,\ \operatorname{Im} \lambda \geq -\gamma\} \leq C_{w,\gamma} \lambda^{\mu},$$

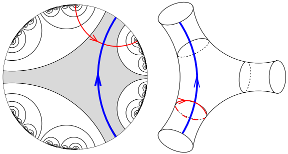
*where* $\mu = \frac{\dim(K)-1}{2}$ *(box dimension).*

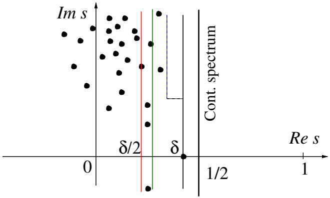
Idea: count how many quantum states $u_\lambda$ can be "hosted" on $K$

1. resonant states are microlocalized in a $\sqrt{h}$-nbhd of $K$ ($h \sim \lambda^{-1}$)
2. Each quantum state occupies a "Planck cell" of volume $\sim h^d$
3. $\rightsquigarrow$ count the number of "Planck cells" in this nbhd.

**Conjecture:** this upper bound is *sharp*, at least for $\gamma$ large enough [LIN-ZWORSKI'04]: *Fractal Weyl's law*.

## Counting long-living resonances for a chaotic $K$






What is the optimal resonance gap? Recall $\gamma_{cl} = -\mathcal{P}(1)$, the local decay of a classical density cloud of points.
Quantum decay rate = classical decay rate if $\operatorname{Im}\lambda_j = -\gamma_{cl}/2$.
Resonances with $\operatorname{Im}\lambda_j > -\gamma_{cl}/2$ are called "supersharp".

- *Conjecture* [JAKOBSON-NAUD] On $\Gamma\backslash\mathbb{H}^2$, there are at most finitely many "supersharp resonances" (at high frequency, $\operatorname{Re} s_j \leq \frac{\delta}{2} + o(1)$). They prove that there are infinitely many resonances with $\operatorname{Re} s_j \geq \frac{\delta(1-2\delta)}{2}$.
- [NAUD, DYATLOV]: the counting of "supersharp resonances" is smaller than the fractal Weyl's law: $\forall\alpha > 0$, $\exists\tau(\alpha) > 0$, $\#\{s_j :\ \operatorname{Re} s_j \geq \delta/2+\alpha,\ |\operatorname{Im} s_j| \leq \lambda\} = \mathcal{O}(\lambda^{\delta+1-\tau(\alpha)})$

# Perspectives

- improve the "pressure bound" in higher dimensions? Resonance gap for thick trapped sets? (the Fractal uncertainy principle works in 2D only)
- *Lower bounds* on the resonance counting are more difficult to obtain (cf. nonselfadjoint spectral problem). Main trick: trace formulae [SJÖSTRAND-ZWORSKI, GUILLOPÉ-ZWORSKI, JAKOBSON-NAUD]
- Structure of the resonant modes $v_j(x)$? At high frequency, they are microlocalized along the unstable manifold of $K$ [BONY-MICHEL,KEATING et al.,N-ZWORSKI]. Can we get local $L^p$ bounds? Difficulty: spectral projectors are not orthogonal.
- adapt to waves on compact domains / compact manifolds with chaotic flow and *nonuniform damping* (cf. V.Petkov's talk)[SCHENCK,ANANTHARAMAN, RIVIÈRE]
- adapt to nonscalar waves?

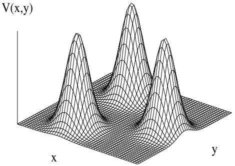
*Merci pour votre attention*

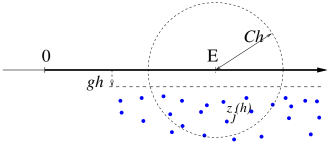
# High frequency / semiclassical regime (2)

The analysis can be generalized to

- scalar waves scattered by nonflat metrics ($g \neq g_0$ in a bounded region of $\mathbb{R}^d$), $\Delta_\Omega \to \Delta_g$
- quantum waves scattered by a potential semiclassical Schrödinger equ.

$$ih\partial_t u = P_h u, \qquad P_h = -h^2\Delta + V(x)$$




The wavepacket $u(t) = e^{-itP_h/h}u_{\rho_0}$ follows the Hamiltonian flow generated by $p(x,\xi) = |\xi|^2 + V(x)$ on $T^*\mathbb{R}^d$.

The resonances $\{z_j(h)\}$ (poles of $\chi(P_h - z)^{-1}\chi$) are associated with metastable state $v_j(h)$ of lifetimes $\frac{h}{|\operatorname{Im} z_j|}$.
$\rightsquigarrow$ focus on the *long living* resonances $\operatorname{Im} z_j = \mathcal{O}(h)$.