Weyl asymptotics for the eigenvalues of dissipative operators and application to scattering

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Outline

1. Dissipative eigenvalues and application to scattering theory
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Dissipative eigenvalues. Let $K \subset \mathbb{R}^d$, $d \geq 2$, be a bounded non-empty domain and let $\Omega = \mathbb{R}^d \setminus \bar{K}$ be connected. We suppose that the boundary Γ of K is C^∞. Consider the boundary problem

$$\begin{cases} u_{tt} - \Delta_x u + c(x) u_t = 0 \text{ in } \mathbb{R}_t^+ \times \Omega, \\ \partial_\nu u - \gamma(x) u_t - \sigma(x) u = 0 \text{ on } \mathbb{R}_t^+ \times \Gamma, \\ u(0, x) = f_0, \; u_t(0, x) = f_1 \end{cases} \tag{1}$$

with initial data $f = (f_1, f_2)$ in the energy space $\mathcal{H} = H^1(\Omega) \times L^2(\Omega)$ with norm

$$\|f\| = \Big(\int_\Omega (|\nabla_x f_1|^2 + |f_2|^2) dx + \int_\Gamma \sigma(x) |f_1|^2 dS_x \Big)^{1/2}.$$

Here ν is the unit outward normal to Γ pointing into Ω, $\gamma(x) \geq 0$, $\sigma(x) \geq 0$ are C^∞ functions on Γ and $0 \leq c(x) \in C_0^\infty(\mathbb{R}^d)$.

The solution of (5) is given by $V(t)f = e^{tG}f,\ t \geq 0$, where $V(t)$ **is a contraction semi-group** in $\mathcal{H}$ whose generator $G = \begin{pmatrix} 0 & 1 \\ \Delta & c \end{pmatrix}$ has a domain $D(G)$ which is the closure in the graph norm of functions $(f_1, f_2) \in C^\infty_{(0)}(\mathbb{R}^n) \times C^\infty_{(0)}(\mathbb{R}^n)$ satisfying the boundary condition $\partial_\nu f_1 - \gamma f_2 - \sigma(x) f_1 = 0$ on Γ. The spectrum of G in $\mathrm{Re}\, z < 0$ is formed by isolated eigenvalues with finite multiplicity. For simplicity in the following we assume that $c(x) = 0, \sigma(x) = 0$. Notice that if $Gf = \lambda f$ with $f = (f_1, f_2) \neq 0$ and $\partial_\nu f_1 - \gamma f_2 = 0$ on Γ, we get

$$\begin{cases} (\Delta - \lambda^2) f_1 = 0 \text{ in } \Omega, \\ \partial_\nu f_1 - \lambda \gamma f_1 = 0 \text{ on } \Gamma. \end{cases} \tag{2}$$

Moreover, $u(t, x) = V(t)f = e^{\lambda t} f(x), \mathrm{Re}\, \lambda < 0$, is a solution of (5) with exponentially decreasing global energy. Such solutions are called asymptotically disappearing and they perturb the scattering. On the other hand, a solution $V(t)f$ is called disappearing if there exists $T > 0$ such that $V(t)f \equiv 0$ for $\forall t \geq T$.

I. It was proved (Colombini, -P. Rauch, (2014)) that if we have a least one eigenvalue λ of G with $\mathrm{Re}\,\lambda < 0$, then the wave operators $W_{\pm}$ are not complete, that is $\mathrm{Ran}\, W_{-} \neq \mathrm{Ran}\, W_{+}$ and we cannot define the scattering operator S by $S = W_{+}^{-1} \circ W_{-}$.

Idea of the proof.

Introduce the spaces

$$H_{+} = \{f \in \mathcal{H} : \ V(t)f \to 0 \,\mathrm{as}\, t \to +\infty\}, \ H_{-} = \{f \in \mathcal{H} : \ V^{*}(t)f \to 0 \,\mathrm{as}\, t \to +\infty\}.$$

First one proves that $\overline{\mathrm{Ran}\, W_{\pm}} = \mathcal{H} \ominus H_{\pm}$. The equality $\mathrm{Ran}\, W_{-} = \mathrm{Ran}\, W_{+}$ yields $H_{+} = H_{-}$. If f is an eigenfunction with eigenvalue $\lambda, \mathrm{Re}\,\lambda < 0$, clearly $f \in H_{+}$. Second, we show that $f \in H_{-}$ implies that $V(t)f$ is disappearing which is impossible. Thus $f \notin H_{-}$. We may define S by using another evolution operator.

II. For problems associated to unitary groups (the global energy is conserved in time) the associated scattering operator $S(z) : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ satisfies

$$S^{-1}(z) = S^*(\bar{z}),\ z \in \mathbb{C},$$

if $S(z)$ is invertible at z. This implies that $S(z)$ is invertible for $\operatorname{Im} z > 0$, since $S(z)$ and $S^*(z)$ are analytic for $\operatorname{Im} z < 0$. For dissipative boundary problems the above relation is not true and $S(z_0)$ may have a non trivial kernel for some $z_0, \operatorname{Im} z_0 > 0$. In this case Lax and Phillips proved that $\mathbf{i}z_0$ is an eigenvalue of G.

It is easy to see that if we have one disappearing solution, then the space

$$H_T = \{f \in \mathcal{H} :\ V(t)f \equiv 0,\ t \geq T\}$$

has infinite dimension. On the other hand, Majda (1975) established that if K and $\gamma(x)$ are analytic, then in the case $\gamma(x) \neq 1,\ \forall x \in \Gamma,$ there are no disappearing solutions. We consider two cases:

$$(\mathbf{A}) :\ 0 < \gamma(x) < 1, \forall x \in \Gamma,\ (\mathbf{B}) :\ \gamma(x) > 1,\ \forall x \in \Gamma.$$

2.Results

Proposition 1 (-P. (2016), (2021))

Let $K = B_3 = \{x \in \mathbb{R}^3 : |x| \leq 1\}$ *and suppose that* $\gamma \equiv const.$ *Then*

(1) $\gamma \equiv 1$. *There are no eigenvalues of* G *in* $\mathbb{C}$.

(2) $\gamma > 1$. *All eigenvalues of* G *are real, we have an infinite number of eigenvalues of* G *and*

$$\sigma_p(G) \subset (-\infty, -\frac{1}{\gamma - 1}].$$

(3) $0 < \gamma < 1$. *The eigenvalues of* G *are not real, we have an infinite number eigenvalues of* G *and*

$$\sigma_p(G) \subset \{\lambda \in \mathbb{C} : |\mathrm{Re}\,\lambda| < 2(1-\gamma)|\mathrm{Im}\,\lambda|^2, \ \mathrm{Re}\,\lambda < 0\}.$$

We see that when $\gamma \searrow 1$ and $\gamma \nearrow 1$ one obtains very large regions without eigenvalues. The result (1) has been anounced by Majda (1975) without proof.

Eigenvalues free regions

Theorem 1 (-P. (2016))

In the case (A) for every ϵ, $0 < \epsilon \ll 1$, the eigenvalues of G lie in the region

$$\Lambda_\epsilon = \{\lambda \in \mathbb{C} : \ |\operatorname{Re}\lambda| \leq C_\epsilon(|\operatorname{Im}\lambda|^{\frac{1}{2}+\epsilon} + 1), \ \operatorname{Re}\lambda < 0\}.$$

In the case (B) for every ϵ, $0 < \epsilon \ll 1$, and every $M \in \mathbb{N}$ the eigenvalues of G lie in the region $\Lambda_\epsilon \cup \mathcal{R}_M$, where

$$\mathcal{R}_M = \{|\operatorname{Im}\lambda| \leq C_M(1 + |\operatorname{Re}\lambda|)^{-M}, \ \operatorname{Re}\lambda < 0\}.$$

For strictly convex obstacles K we improve the above result in the case (B).

Theorem 2 (-P. (2016))

Assume K strictly convex. In the case (B) for every $M \in \mathbb{N}$ the eigenvalues of G lie in the region $\mathcal{R}_M \cup \{|\lambda| < R, \operatorname{Re}\lambda < 0\}$.

By applying the results of Vodev (2017) for the Dirichlet-to-Neumann map, it is possible to improve the above result replacing the region Λ_ϵ by a strip

$$\mathcal{M} = \{\lambda \in \mathbb{C} : -R_0 \leq \mathrm{Re}\,\lambda < 0\}, \; R_0 > 0.$$

Thus for strictly convex obstacles the eigenvalue free regions correspond to the case of a ball.

Previous results have been proved by Majda (1976). He proved that in the case (A) the eigenvalues lie in

$$E_1 = \{\lambda \in \mathbb{C} : |\mathrm{Re}\,\lambda| \leq C_1(|\mathrm{Im}\,\lambda|^{3/4} + 1), \; \mathrm{Re}\,\lambda < 0\},$$

while in the case (B) he showed that the eigenvalues lie in $E_1 \cup E_2$, where

$$E_2 = \{\lambda \in \mathbb{C} : |\mathrm{Im}\,\lambda| \leq C_2(|\mathrm{Re}\,\lambda|^{1/2} + 1), \mathrm{Re}\,\lambda < 0\}.$$

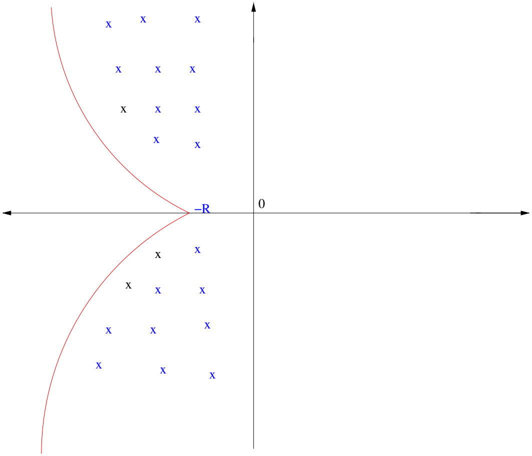


Figure 1: Eigenvalues for $0 < \gamma(x) < 1$

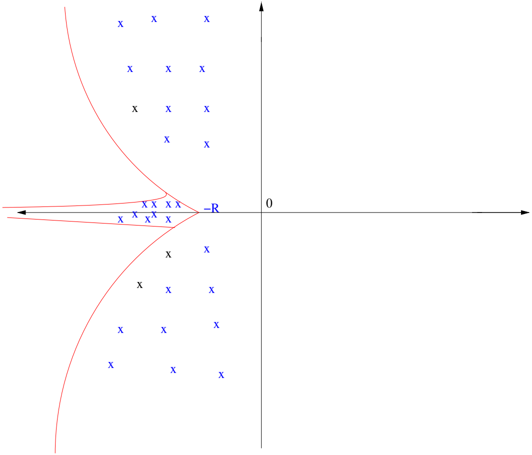


Figure 2: Eigenvalues for $\gamma(x) > 1$

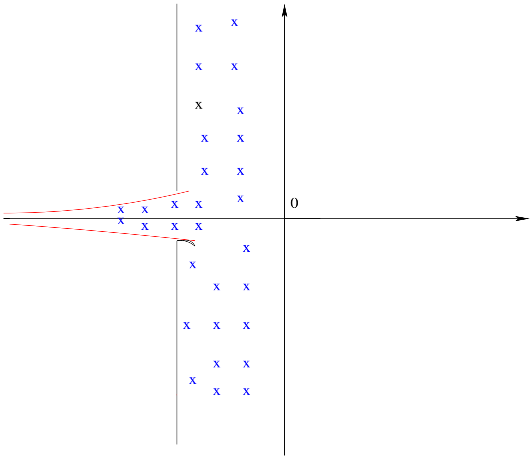


Figure 3: Improved region of eigenvalues for $\gamma(x) > 1$

Weyl asymptotic for the eigenvalues in the case (B)

Introduce the set
$\Lambda = \{\lambda \in \mathbb{C} : \; |\mathrm{Im}\,\lambda| \leq C_2(1+|\mathrm{Re}\,\lambda|)^{-2}, \; \mathrm{Re}\,\lambda \leq -C_0 \leq -1\}, \; \frac{2C_2}{C_0} \leq 1$, containing $\mathcal{R}_M$, $\forall M \geq 2$ modulo compact set. Given $\lambda \in \sigma_p(G)$, we define the algebraic multiplicity of λ by

$$\mathrm{mult}\,(\lambda) = \mathrm{tr}\, \frac{1}{2\pi \mathbf{i}} \int_{|z-\lambda|=\epsilon} (z-G)^{-1} dz$$

with $0 < \epsilon \ll 1$ sufficiently small.

Theorem 3 (-P. (2021))

Assume $\gamma(x) > 1$ for all $x \in \Gamma$. Then the counting function of the eigenvalues in Λ taken with their multiplicities has the asymptotic

$$\begin{aligned} &\sharp\{\lambda_j \in \sigma_p(G) \cap \Lambda : \; |\lambda_j| \leq r, \; r \geq C_\gamma\} \\ &= \frac{\omega_{d-1}}{(2\pi)^{d-1}} \Big(\int_\Gamma (\gamma^2(x) - 1)^{(d-1)/2} dS_x \Big) r^{d-1} + \mathcal{O}_\gamma(r^{d-2}), \; r \to \infty, \end{aligned} \tag{3}$$

ω_{d-1} being the volume of the unit ball $\{x \in \mathbb{R}^{d-1} : \; |x| \leq 1\}$.

Remark 1

For strictly convex obstacles we obtain the asymptotic of all eigenvalues. The constant C_γ depend on γ. When $\min_{x\in\Gamma}\gamma(x) \nearrow 1$, one has $C_\gamma \to +\infty$. This is justified by the proof of Theorem 3 and by the example for the ball B_3.

Remark 2

The behavior of the eigenvalues in the case $0 < \gamma(x) < 1$ is an open problem. In this case the continuation of the exterior Dirichlet-to-Neumann operator $\mathcal{N}(\lambda)$ defined below across the imaginary axis plays an important role. We conjecture that for strictly convex obstacles one has the asymptotic

$$\begin{aligned}&\sharp\{\lambda_j \in \sigma_p(G) \cap \{\lambda \in \mathbb{C} :\ -R_0 \leq \mathrm{Re}\,\lambda < 0,\ |\lambda_j| \leq r,\ r \geq C_\gamma\}\\ &= \frac{\omega_{d-1}}{(2\pi)^{d-1}}\Big(\int_\Gamma (1-\gamma^2(x))^{(d-1)/2} dS_x\Big) r^{d-1} + \mathcal{O}_\gamma(r^{d-2}),\ r \to \infty.\end{aligned} \tag{4}$$

3. Dirichlet-to-Neumann map and trace formula

For $\operatorname{Re}\lambda < 0$ introduce the exterior Dirichlet-to-Neumann map

$$\mathcal{N}(\lambda) : H^s(\Gamma) \ni f \longrightarrow \partial_\nu u|_\Gamma \in H^{s-1}(\Gamma),$$

where u is the solution of the problem

$$\begin{cases} (-\Delta + \lambda^2)u = 0 \text{ in } \Omega, \ u \in H^2(\Omega), \\ u = f \text{ on } \Gamma, \\ u : (\mathbf{i}\lambda) - \text{outgoing}. \end{cases} \tag{5}$$

A function $u(x)$ is $(\mathbf{i}\lambda)$-outgoing if there exists $R > \rho_0$ and $g \in L^2_{comp}(\mathbb{R}^d)$ such that

$$u(x) = (-\Delta_0 + \lambda^2)^{-1} g, \ |x| \geq R,$$

where $R_0(\lambda) = (-\Delta_0 + \lambda^2)^{-1}$ is the outgoing resolvent of the free Laplacian $-\Delta_0$ in $\mathbb{R}^d$ which is analytic in $\mathbb{C}$ for d odd and on the logarithmic covering of $\mathbb{C}$ for d even.

The operator $\mathcal{N}(\lambda)$ can be expressed by the cut-off resolvent $\chi(-\Delta_D + \lambda^2)^{-1}\chi$ of the Dirichlet Laplacian Δ_D, hence $\mathcal{N}(\lambda)$ is analytic in $\{\lambda : \mathrm{Re}\,\lambda < 0\}$. The boundary condition for an eigenfunction g becomes

$$\mathcal{C}(\lambda)f := \mathcal{N}(\lambda)f - \lambda\gamma f = 0,\ f = g|_\Gamma.$$

The operator $\mathcal{N}(\lambda) :\ H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is compact and invertible in $\{z : \mathrm{Re}\,\lambda < 0\}$ since there are no resonances of the Neumann problem in $\{z : \mathrm{Re}\,z < 0\}$. We write

$$\mathcal{C}(\lambda) = (Id - \lambda\gamma\mathcal{N}(\lambda)^{-1})\mathcal{N}(\lambda).$$

and by Fredholm theorem one deduces that $\mathcal{C}(\lambda)^{-1}$ is meromorphic in $\{\lambda : \mathrm{Re}\,\lambda < 0\}$.

Trace formula

Proposition 2

Let $\alpha \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$ be a closed positively oriented curve without self intersections. Assume that $\mathcal{C}(\lambda)^{-1}$ has no poles on α . Then

$$\operatorname{tr}_{\mathcal{H}} \frac{1}{2\pi i} \int_\alpha (\lambda - G)^{-1} d\lambda = \operatorname{tr}_{H^{1/2}(\Gamma)} \frac{1}{2\pi i} \int_\alpha \mathcal{C}(\lambda)^{-1} \frac{\partial \mathcal{C}}{\partial \lambda}(\lambda) d\lambda. \tag{6}$$

Since G has only point spectrum in $\operatorname{Re}\lambda < 0$, the left hand term in (6) is equal to the number of the eigenvalues of G in the domain ω bounded by α counted with their algebraic multiplicities. Setting $\tilde{\mathcal{C}}(\lambda) = \frac{\mathcal{N}(\lambda)}{\lambda} - \gamma$, we write the right hand side of (6) as

$$\operatorname{tr} \frac{1}{2\pi i} \int_\alpha \tilde{\mathcal{C}}(\lambda)^{-1} \frac{\partial \tilde{\mathcal{C}}}{\partial \lambda}(\lambda) d\lambda. \tag{7}$$

Set $\lambda = -\frac{1}{\tilde{h}}$, $0 < \operatorname{Re} \tilde{h} \ll 1$ and consider the problem

$$\begin{cases} (-\tilde{h}^2\Delta + 1)u = 0 \text{ in } \Omega, \\ -\tilde{h}\partial_\nu u - \gamma u = 0 \text{ on } \Gamma, \\ u - \text{outgoing}. \end{cases} \tag{8}$$

We introduce the operator $C(\tilde{h}) := -\tilde{h}\mathcal{N}(-\tilde{h}^{-1}) - \gamma$ and using (7), the trace formula (6) becomes

$$\operatorname{tr} \frac{1}{2\pi i} \int_\alpha (\lambda - G)^{-1} d\lambda = \operatorname{tr} \frac{1}{2\pi i} \int_{\tilde{\alpha}} C(\tilde{h})^{-1} \dot{C}(\tilde{h}) d\tilde{h}, \tag{9}$$

where $\dot{C}$ denote the derivative with respect to $\tilde{h}$ and $\tilde{\alpha}$ is the curve $\tilde{\alpha} = \{z \in \mathbb{C} : z = -\frac{1}{w},\ w \in \alpha\}$.

Recall that $\Lambda = \{\lambda \in \mathbb{C} : |\operatorname{Im}\lambda| \leq C_2(|\operatorname{Re}\lambda| + 1)^{-2}, \operatorname{Re}\lambda \leq -C_0 \leq -1\}$. For $\lambda \in \Lambda$ one has $|\operatorname{Im}\lambda| \leq 1$ and this implies $\tilde{h} \in L$, where

$$L := \{\tilde{h} \in \mathbb{C} : |\operatorname{Im}\tilde{h}| \leq C_1|\tilde{h}|^4, |\tilde{h}| \leq C_0^{-1}, \operatorname{Re}\tilde{h} > 0\}. \tag{10}$$

We write the points in L as $\tilde{h} = h(1 + \mathbf{i}\eta)$ with $0 < h \leq h_0 \leq C_0^{-1}$, $\eta \in \mathbb{R}$, $|\eta| \leq h^2$. Therefore the problem (8) becomes

$$\begin{cases} (-h^2\Delta - z)u = 0 \text{ in } \Omega, \\ -(1 + \mathbf{i}\eta)h\partial_\nu u - \gamma u = 0 \text{ on } \Gamma, \\ u - \text{outgoing}. \end{cases} \tag{11}$$

with $z = -\frac{1}{(1+\mathbf{i}\eta)^2} = -1 + s(\eta)$, $|s(\eta)| \leq (2 + h^2)h^2 \leq 3h^2$.

Semi-classical parametrix

Given $f \in H^s(\Gamma)$, consider the problem

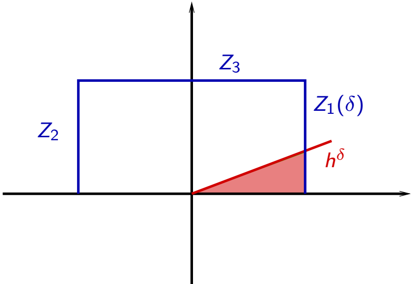
$$\begin{cases} (-h^2\Delta - z)u = 0 \text{ in } K, \\ u = f \text{ on } \Gamma. \end{cases} \tag{12}$$

Let $z \in Z_1 \cup Z_2 \cup Z_3$ and $\lambda = \mathbf{i}\frac{\sqrt{z}}{h}$, where

$$Z_1 = \{\mathrm{Re}\, z = 1, 0 \leq |\,\mathrm{Im}\, z| \leq 1\},\ Z_1(\delta) = Z_1 \cap \{|\,\mathrm{Im}\, z| \geq h^\delta\},$$

$$Z_2 = \{\mathrm{Re}\, z = -1, 0 \leq |\,\mathrm{Im}\, z| \leq 1\},\ Z_3 = \{|\,\mathrm{Re}\, z| \leq 1, |\,\mathrm{Im}\, z| = 1\}.$$

Figure 4: Contours $Z_1(\delta), Z_2, Z_3$



Let γ_0 denote the trace on Γ. Consider the problem (12) for $z \in Z_1(1/2 - \epsilon) \cup Z_2 \cup Z_3$ with $0 < \epsilon \ll 1$ and define the semi-classical interior Dirichlet-to-Neumann map

$$\mathcal{N}_{int}(z, h) : H_h^s(\Gamma) \ni f \longrightarrow -\mathbf{i}\gamma_0 h\partial_\nu u \in H_h^{s-1}(\Gamma).$$

Here $H_h^s(\Gamma)$ is the semi-classical Sobolev space with norm $\|\langle hD\rangle^s u\|_{L^2(\Gamma)}$. G. Vodev (2015) constructed for domains with arbitrary geometry a semi-classsical parametrix for (7) as a FIO with complex phase $\varphi(x, \xi'; z)$ in a small neighborhood of the boundary Γ. Close to the boundary introduce geodesic normal coordinates (x', x_d) in a neighborhood of a point $x_0 \in \Gamma$ with $x_d = 0$ on Γ(we take $x_d = \mathrm{dist}(x, \Gamma)$). The eikonal equation and the transport equations can be solved only modulo $\mathcal{O}(x_d^N)$, $\forall N \gg 1$.

Set $x = (x', x_d)$, $\xi = (\xi', \xi_d)$. We say that $a(x', \xi'; h) \in S^k_\delta(\Gamma)$ if the following conditions are satisfied:

$$|\partial_x'^{\alpha} \partial_{\xi'}^{\beta} a(x, \xi'; h)| \leq C_{\alpha,\beta} h^{-\delta(|\alpha|+|\beta|)} \langle \xi' \rangle^{k-|\beta|}, \ \forall \alpha, \forall \beta,$$

where $\langle \xi' \rangle = (1 + |\xi'|^2)^{1/2}$. For $a \in S^k_\delta(\Gamma)$, we consider the operator

$$\Big(Op_h(a)f\Big)(x) = (2\pi h)^{-d+1} \int\int e^{i\langle x'-y', \xi' \rangle/h} a(x, \xi'; h) f(y') dy d\xi'.$$

We have a calculus for the $h-$pseudo-differential operators with symbols in S^k_δ if $0 < \delta < 1/2$. The semiclassical symbol of $-h^2\Delta$ becomes $\xi_d^2 + r(x, \xi') + hq(x)\xi_d$ and $r(x', 0, \xi') = r_0(x', \xi')$ is the principal symbol of the Laplace-Beltrami operator $-h^2\Delta|_\Gamma$ on Γ.

For $z \in Z_1 \cup Z_2 \cup Z_3$, let

$$\rho(x', \xi', z) = \sqrt{z - r_0(x', \xi')} \in C^\infty(T^*(\Gamma)), \ \mathrm{Im}\, \rho > 0$$

be the root of the equation $\rho^2 + r_0(x', \xi') - z = 0$. It is easy to see that $\rho \in S^1_{1/2-\epsilon}$, if $z \in Z_1(1/2 - \epsilon)$, $\rho \in S^1_0$, if $z \in Z_2 \cup Z_3$.

Proposition 3 (Vodev, (2015))

Given $0 < \epsilon \ll 1$, there exists $0 < h_0(\epsilon) \ll 1$ such that for $z \in Z_1(1/2 - \epsilon)$ and $0 < h \leq h_0(\epsilon)$ we have

$$\|\mathcal{N}(z, h) - Op_h(\rho + hb)\|_{L^2(\Gamma) \to H^1_s(\Gamma)} \leq \frac{Ch}{\sqrt{|\mathrm{Im}\, z|}}, \tag{13}$$

where $C > 0$ is independent of h, z, ϵ and $b \in S^0_0$ does not depend on z, h. Moreover, for $z \in Z_2 \cup Z_3$ the above estimate holds with $|\mathrm{Im}\, z|$ replaced by 1.

Exterior Dirichlet-to-Neumann map

For our analysis we need to apply the exterior DIrichlet-to-Neumann map

$$\mathcal{N}_{ext}(z,h) : H_h^s(\Gamma) \ni f \longrightarrow -\mathbf{i}\gamma_0 h\partial_\nu u \in H_h^{s-1}(\Gamma),$$

where u is the outgoing solution of the problem

$$(-h^2\Delta - z)u = 0 \text{ in } \Omega = \mathbb{R}^d \setminus \bar{K},\ u|_\Gamma = f.$$

The operator $\mathcal{N}_{ext}(z,h)$ is a meromorphic function related to the cut-off outgoing resolvent $\chi(h^2 G_D - z)^{-1}\chi$ with poles in the half-plane $\{\mathrm{Im}\, z < 0\}$. A result completely analogous to (13) was proved by -P. (2016). For strictly convex obstacles K and $\mathrm{Re}\, z \sim 1,\ |\,\mathrm{Im}\, z| \leq c_0 h^{2/3}$ Sjöstrand (2014) obtained results similar to Prop. 3. The case $h^{1/2-\epsilon} \leq \mathrm{Im}\, z \leq c_0 h^{2/3}$ for strictly convex obstacles has been covered by -P. (2016) by a semi-classical parametrix construction inspired by that of Vodev.

4. Idea of the proof of Theorem 3

We use a parametrix $T(z,h)$ for $\mathcal{N}_{ext}(z,h) = \mathcal{N}(z,h)$ for $z = -1 + s(\eta)$, $|s(\eta)| \leq h^2$ such that

$$\|\mathcal{N}(z,h)f - T(z,h)f\|_{H_h^m(\Gamma)} \leq C_{m,N} h^{-s_d+N} \|f\|_{L^2(\Gamma)},\ \forall N \in \mathbb{N}. \tag{14}$$

Notice that $\mathcal{N}(-1,h)$ is self-adjoint. Introduce the self-adjoint operator

$$P(h) := T(-1,h) - \gamma(x'),\ 0 < h \leq h_0.$$

The semiclassical principal symbol of $P(h)$ is
$p_1(x',\xi') = -\mathbf{i}\sqrt{-1 - r_0} - \gamma(x') = \sqrt{1 + r_0} - \gamma(x')$. Since $\min_{x'} \gamma(x') > 1$, this symbol vanishes when

$$r_0(x',\xi') = \gamma^2(x') - 1 > 0.$$

We will treat $P(h)$ as a classical pdo with symbol

$$\sqrt{1 + h^2 r_0(x',\xi')} - \gamma(x) + P_0(h),\ P_0(h) \in S^0.$$

We apply the approach of Sjöstrand-Vodev (1997) concerning the asymptotic of Rayleigh resonances close to the real axis. Let

$$\mu_1(h) \leq \mu_2(h) \leq \ldots \leq \mu_m(h) \leq \ldots$$

be the eigenvalues of $P(h)$ counted with their multilipcities. The points $0 < h_k \leq h_0$, where $\mu_k(h_k) = 0$ correspond to points for which $P(h)$ is not invertible. For large fixed k_0, depending on h_0, the eigenvalues $\mu_k(h_0)$ are positive, whenever $k > k_0$. Thus if $\mu_k(r^{-1}) < 0,\ k > k_0$ and $r > h_0^{-1}$, we have $\mu_k(h_k) = 0$ for some $r^{-1} < h_k < h_0$. However, a more precise analysis of the behaviour of $\mu_k(h)$ and the relation of h_k to eigenvalues $\lambda_j \in L$ of G is necessary. Thus the problem is reduced to a Weyl asymptotic of the counting function of the negative eigenvalues of $P(r^{-1}),\ r \geq C_\gamma = (h_0(\gamma))^{-1}$ given by the well known formula

$$\frac{r^{d-1}}{(2\pi)^{d-1}} \iint_{r_0(x',\xi') \leq \gamma^2(x')-1} dx' d\xi' + \mathcal{O}_\gamma(r^{d-2}). \tag{15}$$

Main steps

1. Examine $\frac{dP(h)}{dh}$ and $\frac{d\mu_k(h)}{dh}$ and prove that the zero h_k of $\mu_k(h)$ is unique.
2. Study the continuation $P(\tilde{h})$ for $\tilde{h} = h(1 + \mathbf{i}\eta)$, $|\eta| \leq h^2$ and show that

$$\|P^{-1}(\tilde{h})\|_{\mathcal{L}(H^s, H^{s+1})} \leq C_s \frac{1}{|\eta|}, \ \eta \neq 0. \tag{16}$$

3. Establish a trace formula

$$\mathrm{tr}_{H^{1/2}(\Gamma)} \frac{1}{2\pi\mathbf{i}} \int_{\gamma_{k,p}} P^{-1}(\tilde{h}) \frac{dP(\tilde{h})}{d\tilde{h}} d\tilde{h}$$

with suitable curve $\gamma_{k,p}$ counting the number of h_k in a domain bounded by $\gamma_{k,p}$.

4. Show that the trace formulas for $C(\tilde{h})$ and $P(\tilde{h})$ over $\gamma_{k,p}$ differ by a negligible term $\mathcal{O}_m(h^p)$, $\forall p \in \mathbb{N}$. Thus we obtain a map $\ell: \ h_k \to \ell(h_k) = \lambda_k$ between the set of points $h_k \in]0, h_0]$ and the eigenvalues $\lambda_k \in L$.

Idea for the Step 1

Set $\min_{x\in\Gamma}\gamma(x) = c_0 > 1$, $\max_{x\in\Gamma}\gamma(x) = c_1 \geq c_0$ and choose a constant $C = \frac{2}{c_1^2}$. We denote by $(.,.)$ the scalar product in $L^2(\Gamma)$ and for two self adjoint operators L_1, L_2 the inequality $L_1 \geq L_2$ means $(L_1 u, u) \geq (L_2 u, u)$, $\forall u \in L^2(\Gamma)$.

Proposition 4

Let $\langle h\Delta\rangle = (1 - h^2\Delta_\Gamma)^{1/2}$ and let $\epsilon = C(c_0 - 1)^2 < 2$. Then for h sufficiently small we have

$$h\frac{\partial P(h)}{\partial h} + CP(h)\langle h\Delta\rangle^{-1/2}P(h) \geq \epsilon\Big(1 - \frac{C_2}{\epsilon}h\Big)\langle h\Delta\rangle \tag{17}$$

with a constant $C_2 > 0$ independent of h and ϵ.

Remark 3

The values of ϵ depends on $(c_0 - 1)^2$ and $\epsilon \searrow 0$ when $c_0 \searrow 1$. Also $0 < h < \frac{\epsilon}{C_2}$ so h_0 and $h_k \in]0, h_0]$ must have order $o(\epsilon)$. Hence we need to take $r \geq \frac{1}{o(\epsilon)}$ in (15).

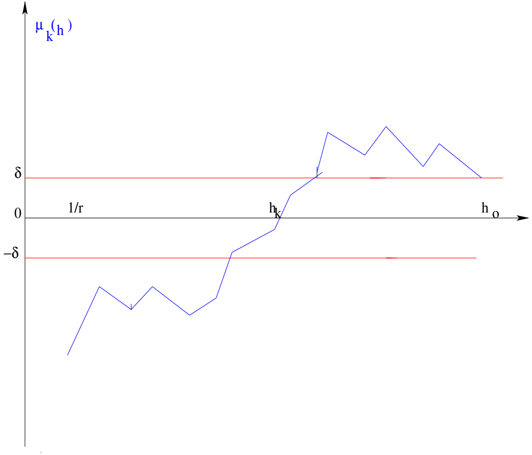


Figure 5: eigenvalue $\mu_k(h)$ for $1/r \leq h \leq h_o$

Let h_1 be small and let $\mu_k(h_1)$ have multiplicity m. For h close to h_1 one has exactly m eigenvalues and we denote by $F(h)$ the space spanned by them. We can find a small interval (α, β) around $\mu_k(h_1)$, independent on h, containing the eigenvalues spanning $F(h)$. Given $h_2 > h_1$ close to h_1, consider a normalised eigenfunction $e(h_2)$ with eigenvalue $\mu_k(h_2)$. Denote by dot the derivative with respect to h. Let $\pi(h) = E_{(\alpha,\beta)}$ be the spectral projection of $P(h)$, hence $F(h) = \pi(h)L^2(\Gamma)$. Then $(\pi(h) - I)\pi(h) = 0$ yields $\pi(h)\dot{\pi}(h)\pi(h) = 0$ and $\dot{\pi}(h)|_{F(h)} = 0$. We construct a smooth extension $e(h) = \pi(h)e(h_2) \in F(h),\ h \in [h_1, h_2]$ of $e(h_2)$ with $\|e(h)\| = 1,\ \dot{e}(h) \in F(h)^{\perp}$. Obviously, $e(h_1)$ will be normalised eigenfunction with eigenvalue $\mu_k(h_1)$. One obtains

$$h\dot{P}(h) = h^2\Delta\langle hD\rangle^{-1} + hL_0 = P(h) - \langle hD\rangle^{-1} + hL_1$$

with zero order operators L_0, L_1 and this implies $|(\dot{P}(h)e(h), e(h))| \leq C_0 h^{-1},\ h \in [h_1, h_2]$. Therefore

$$|\mu_k(h_2) - \mu_k(h_1)| = \Big|\int_{h_1}^{h_2} \frac{d}{dh}(P(h)e(h), e(h))dh\Big| \leq C_0 \int_{h_1}^{h_2} h^{-1}dh \leq \frac{C_0}{h_1}(h_2 - h_1).$$

Assuming $\mu_k(h) \in [-\delta, \delta]$ for $h \in [h_1, h_2]$, we deduce that $\mu_k(h)$ is locally Lipschitz function in h and its almost defined derivative satisfies $|h\frac{\partial \mu_k(h)}{\partial h}| \leq C_0$.

To estimate $h\frac{\partial \mu_k(h)}{\partial h}$ from below, we exploit Proposition 4. For $h \leq h_0 \leq \frac{\epsilon}{8C_2}$ with $C_1 = C_2/\epsilon$ we have

$$h\frac{\partial \mu_k(h)}{\partial h} = (h\dot{P}(h)e(h), e(h))$$

$$\geq \epsilon(1 - C_1 h)(\langle hD \rangle e(h), e(h)) - C(\langle hD \rangle^{-1} P(h)e(h), P(h)e(h))$$

$$\geq \epsilon(1 - C_1 h) - C\delta^2 \geq \frac{3\epsilon}{4},$$

choosing $\frac{c_0 - 1}{2} > \delta = (c_0 - 1)\sqrt{\frac{1}{4} - C_1 h_0} \geq \frac{(c_0-1)}{2\sqrt{2}}$. Consequently, for $h \in [h_1, h_2]$ one has

$$\mu_k(h_2) - \mu_k(h_1) \geq \frac{3\epsilon}{4} \int_{h_1}^{h_2} h^{-1} dh \geq \frac{3\epsilon}{4h_2}(h_2 - h_1)$$

and we obtain $\frac{3\epsilon}{4} \leq h\frac{d\mu_k(h)}{dh} \leq C_0$.

We fix $c_0 = \frac{3\epsilon}{4}$ and $h_0 > 0$. Let $p > d$ be fixed and let

$$I_{k,p} = \{h \in \mathbb{R} : |h - h_k| \leq \frac{h^{p+1}}{c_0}\}.$$

Then for $h \in]0, h_0] \setminus I_{k,p}$ one has $|\mu_k(h)| \geq h^p$. Thus for $h \in]0, h_0] \setminus \left(\bigcup_{k \geq k_0} I_{k,p}\right)$ one obtains

$$\|P(h)^{-1}\|_{L^2 \to L^2} = \mathcal{O}(h^{-p}). \tag{18}$$

On the other hand, $\sum_{k \geq k_0} |I_{k,p}| = \mathcal{O}(h^{p+2-d})$. We can construct disjoint intervals $J_{k,p}$ so that the estimate (18) holds for $h \in]0, h_0] \setminus \left(\bigcup_{k \geq k_0} J_{k,p}\right)$ with $|J_{k,p}| = \mathcal{O}(h^{p+2-d})$. We choose a curve $\gamma_{k,p} \subset \mathbb{C}$ bounded by four segments

$$\mathrm{Re}\,\tilde{h} \in \partial J_{k,p},\ \mathrm{Im}\,\tilde{h} = \pm \mathrm{Re}\,\tilde{h}^{p+1}.$$

Next we extend the estimate (18) to

$$\|P(\tilde{h})^{-1}\|_{L^2 \to L^2} = \mathcal{O}((\mathrm{Re}\,\tilde{h})^{-p}),\ \tilde{h} \in \gamma_{k,p}. \tag{19}$$

THANK YOU !