# Logarithmic stabilization of an acoustic system with a damping term

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# INTRODUCTION

Model of sound propagation in a fluid. In the following equation
• $r$ a scalar, is the fluctuation of pressure at $(x, t)$ around a fixed pressure $p_0$.
• $u$ valued in $\mathbb{R}^d$ is the particle velocity at $(x, t)$.

$$\begin{cases} u_t + \nabla r + bu = 0, \text{ in } \Omega \times \mathbb{R}^+, \\ r_t + \operatorname{div} u = 0, \text{ in } \Omega \times \mathbb{R}^+, \\ u \cdot n = 0, \text{ on } \Gamma \times \mathbb{R}^+, \\ u(0, x) = u^0(x), \ r(0, x) = r^0(x), \ x \in \Omega, \end{cases}$$

$\Omega$ smooth domain in $\mathbb{R}^d$.
$b$ is a nonnegative function, b is called the *damping*.
We assume $b(x) \geq b_0 > 0$ in $\omega \subset \Omega$.
Energy is defined by $E(t) = \frac{1}{2}\left(\|u(t)\|^2 + \|r(t)\|^2\right)$.

THE GOAL IS TO PROVE THAT ENERGY DECAY TO 0.

# FUNCTIONAL CONTEXT

$L^2_m(\Omega) = \{f \in L^2(\Omega) : \int_\Omega f(x)\,dx = 0\}$.
Let $H = (L^2(\Omega))^d \times L^2_m(\Omega)$,
We introduce the operators

$$\mathcal{A} = \begin{pmatrix} 0 & \nabla \\ \mathrm{div} & 0 \end{pmatrix}$$

$$\mathcal{D}(\mathcal{A}) = \left\{(u,r) \in H,\ (\nabla r, \mathrm{div}\, u) \in H,\ u \cdot n_{|\Gamma} = 0\right\}$$

$$\mathcal{B} = \begin{pmatrix} \sqrt{b} \\ 0 \end{pmatrix} \mathcal{B}^* = \begin{pmatrix} \sqrt{b} & 0 \end{pmatrix}$$

For $u \in (L^2(\Omega))^d$ with $\mathrm{div}\, u \in L^2(\Omega)$, $u \cdot n_{|\Gamma}$ make sens in $H^{-1/2}(\Gamma)$
The problem may be recasted in the form:

$$\begin{cases} Z_t(t) + \mathcal{A}Z(t) + \mathcal{B}\mathcal{B}^* Z(t) = 0,\ t > 0, \\ Z(0) = Z^0, \end{cases} \quad \text{or} \begin{cases} Z_t(t) = \mathcal{A}_d Z(t),\ t > 0, \\ Z(0) = Z^0, \end{cases}$$

where $Z = (u, r)$,
$\mathcal{A}_d = -\mathcal{A} - \mathcal{B}\mathcal{B}^*$ with $\mathcal{D}(\mathcal{A}_d) = \mathcal{D}(\mathcal{A})$.

# ENERGY

Introducing the following energy,

$$E(t) = \frac{1}{2} \, \|(u(t), r(t))\|_H^2 \, , \; \forall \, t \geq 0,$$

we have

$$E(0) - E(t) = \int_0^t \left\| \sqrt{b} \, u(s) \right\|_{(L^2(\Omega))^d}^2 ds, \text{ for all } t \geq 0.$$

Energy is not increasing.

Our goal is to study the decay.

This kind of problem is studied in this last years and is related with the associated stationary problem.

# DECAY AND RELATED PROBLEM

We have several results relating decay and resolvent. We assume the two conditions on semigroup.

1) $\exists M > 0$, $\|e^{t\mathcal{A}_d}\|_{\mathcal{L}(H)} \leq M$, for $t \geq 0$,
2) $\mathcal{A}_d + i\mu I$, is invertible for any $\mu \in \mathbb{R}$,

Stabilisation exponential is implied by the following properties (Gearhart-Huang-Prüss theorem)

$$\|(\mathcal{A}_d + i\mu I)^{-1}\|_{\mathcal{L}(H)} \leq M \Leftrightarrow \|e^{t\mathcal{A}_d}\|_{\mathcal{L}(H)} \leq Ce^{-\delta t}.$$

$$\|(\mathcal{A}_d + i\mu)^{-1}\|_{\mathcal{L}(H)} \leq Ce^{C|\mu|} \Leftrightarrow \|e^{t\mathcal{A}_d}Z\| \leq \frac{C'\|Z\|_{\mathcal{D}(\mathcal{A})}}{\log(3+t)}.$$

$$\|(\mathcal{A}_d + i\mu)^{-1}\|_{\mathcal{L}(H)} \leq C(1+|\mu|)^{\alpha} \Leftrightarrow \|e^{t\mathcal{A}_d}Z\| \leq \frac{C'\|Z\|_{\mathcal{D}(\mathcal{A})}}{(1+t)^{1/\alpha}}.$$

Two last results was obtained by several authors, Burq, Batty-Duyckaerts, Borichev-Tomilov...

# SOME OBSERVATIONS I

Considering the system

$$\begin{cases} u_t + \nabla r = 0, \\ r_t + \operatorname{div} u = 0, \end{cases}$$

We have $r_{tt} - \Delta r = 0$.

That system is related with a wave equation.

Now the damped system

$$\begin{cases} u_t + \nabla r + bu = 0, \\ r_t + \operatorname{div} u = 0, \end{cases}$$

means $r_{tt} - \Delta r - \operatorname{div}(bu) = 0$.

It seems difficult to exploit the link between an acoustic system and a wave equation.
Other observation, the functional spaces are not the same.

# SOME OBSERVATIONS II

The stationary system is the following

$$\begin{cases} -\nabla r + i\mu u - bu = f, \\ -\operatorname{div} u + i\mu r = h, \end{cases}$$

where $(f, h) \in H$. In particular if $\mu = 0$ and $(f, h) = (0, 0)$, we have non trivial solutions to

$$\begin{cases} -\nabla r - bu = 0, \\ \operatorname{div} u = 0, \end{cases}$$

In particular $r = 0$, $u = 0$ on support of $b$ and $\operatorname{div} u = 0$.

It is easy to find solutions to $\operatorname{div} u = 0$, for instance $u_j(x) = \phi'(x_1) g_j(x_2, \ldots, x_d)$ for $j = 2, \ldots, d$ and $u_1(x) = -\phi(x_1) \sum_{j=2}^{d} \partial_{x_j} g_j(x_2, \ldots, x_d)$.

# RESULT FORMULATION

Due to the large kernel, we work on the following functional space $H_0 = \ker[\mathcal{A}_d]^{\perp}$.
If $(r, u) \in H$ we can write $(r, u) = (r_0, u_0) + (r_1, u_1)$ where $(r_0, u_0) \in \ker[\mathcal{A}_d]^{\perp}$ and $(r_1, u_1) \in \ker \mathcal{A}_d$.

With these setting, we have $e^{t\mathcal{A}_d}(r, u) = (r_1, u_1) + e^{t\mathcal{A}_d}(r_0, u_0)$.

The goal is to prove the following result.

There exists $C > 0$ such that

$$E(t) \leq \frac{C\|Z\|^2_{\mathcal{D}(\mathcal{A})}}{\log^2(3+t)},$$

where $Z = (r_0, u_0)$, $\|Z\|_{\mathcal{D}(\mathcal{A})} = \|Z\| + \|\mathcal{A}Z\|$, and $E(t) = \frac{1}{2}\|e^{t\mathcal{A}_d}(r_0, u_0)\|^2_H$.

Observe that only for $\mu = 0$ we have the problem that $\mathcal{A}_d$ is not invertible. For $\mu \neq 0$, we can work on $H$.

# ESTIMATION ON SUPPORT OF $b$

Let $(f, h) \in H$ we have to estimate $(u, r)$ solutions of

$$\begin{cases} -\nabla r + i\mu u - bu = f \\ -\operatorname{div} u + i\mu r = h \end{cases}$$

Taking the inner product, the first equation by $u$ and the second by $r$, we have

$$\begin{aligned} &-(\nabla r|u) + i\mu\|u\|^2 - \|\sqrt{b}u\|^2 = (f|u) \\ &-(r|\operatorname{div} u) - i\mu\|r\|^2 = (r|h) \end{aligned}$$

As $(r|\operatorname{div} u) = (-\nabla r|u)$ with boundary condition $u \cdot n = 0$. Summing we have

$$i\mu\big(\|u\|^2 - \|r\|^2\big) - \|\sqrt{b}u\|^2 = (f|u) - (r|h).$$

Real part yields

$$\int_\omega |u|^2 \lesssim \big(\|f\| + \|h\|\big)\big(\|u\| + \|r\|\big),$$

where $b \geq \delta > 0$ on $\omega$.

# HOW TO CONCLUDE

Assuming we have an estimate of the following form

$$\|u\|^2 + \|r\|^2 \lesssim e^{C\mu}\big(\|f\|^2 + \|h\|^2 + \int_\omega |u|^2\big).$$

With the estimate of the previous slide

$$\int_\omega |u|^2 \lesssim \big(\|f\| + \|h\|\big)\big(\|u\| + \|r\|\big),$$

we obtain

$$\|u\|^2 + \|r\|^2 \lesssim e^{C\mu}\Big(\|f\|^2 + \|h\|^2 + \big(\|f\| + \|h\|\big)\big(\|u\| + \|r\|\big)\Big).$$

Yielding

$$\|u\|^2 + \|r\|^2 \le C' e^{C'\mu}\big(\|f\|^2 + \|h\|^2\big) + \frac{1}{2}\big(\|u\|^2 + \|r\|^2\big).$$

# INTERIOR ESTIMATE I

$$\begin{cases} -\nabla r + i\mu u - bu = f, \\ -\operatorname{div} u + i\mu r = h. \end{cases}$$

Taking the divergence of the first equation we have

$$-\Delta r + i\mu \operatorname{div} u - \operatorname{div}(bu) = \operatorname{div} f,$$

then

$$-\Delta r - \mu^2 r = \operatorname{div} f + i\mu h + \operatorname{div}(bu).$$

One problem is $\operatorname{div} f$ as $f$ is only in $L^2(\Omega)$.
We use Carleman estimate proved for instance by Imanuvilov-Puel

$$-\Delta v - \mu^2 v = G + \operatorname{div} F.$$

$$\tau^3 \|e^{\tau\varphi} v\|^2 + \tau \|e^{\tau\varphi} \nabla v\|^2 \leq C\tau^2 \left( \|e^{\tau\varphi} G\|^2 + \|e^{\tau\varphi} F\|^2 \right).$$

for all $v \in \mathscr{C}_c^\infty(K)$ which satisfies $\tau \geq \tau_*$ and $\mu$ satisfying $c_0\tau \leq |\mu| \leq c_0'\tau$.

# INTERIOR ESTIMATE II

Assuming $r$ and $u$ compactly supported in $\Omega$ and satisfying

$$\begin{cases} -\nabla r + i\mu u - bu = f, \\ -\operatorname{div} u + i\mu r = h. \end{cases}$$

We have

$$-\Delta r - \mu^2 r = \operatorname{div} f + i\mu h + \operatorname{div}(bu).$$

Applying previous Carleman estimate to $r$, we obtain this estimate

$$\tau^3\|e^{\tau\varphi}r\|^2+\tau\|e^{\tau\varphi}\nabla r\|^2 \leq C\tau^2\left(|\mu|^2\|e^{\tau\varphi}h\|^2 + \|e^{\tau\varphi}f\|^2 + \|e^{\tau\varphi}u\|^2\right).$$

First equation in system yields $i\mu u = \nabla r + bu + f$.
From that we can estimate $u$

$$\tau|\mu|^2\|e^{\tau\varphi}u\|^2 \leq C\tau^2\left(|\mu|^2\|e^{\tau\varphi}h\|^2 + \|e^{\tau\varphi}f\|^2 + \|e^{\tau\varphi}u\|^2\right).$$

We can absorb the $u$ term of the right to obtain

$$\begin{aligned} \tau^3\|e^{\tau\varphi}r\|^2 + \tau\|e^{\tau\varphi}\nabla r\|^2 + \tau|\mu|^2\|e^{\tau\varphi}u\|^2 & \\ \leq C\tau^2\left(|\mu|^2\|e^{\tau\varphi}h\|^2 + \|e^{\tau\varphi}f\|^2\right). \end{aligned}$$

# DIFFICULTY FOR BOUNDARY ESTIMATE

We can gluing local Carleman estimate to obtain global Carleman estimate.

Maybe we can follow the approach applied for interior estimate but there is a difficulty.

The boundary condition is imposed on $u$. On $r$ by equation $-\nabla r + i\mu u - bu = f$, we have $-\partial_n r_{|\partial\Omega} = (f \cdot n)_{|\partial\Omega}$.

As $f \in L^2(\Omega)$, it is not clear that $(f \cdot n)_{|\partial\Omega}$ make sense.

To avoid this difficulty we work directly on the system.

# PROBLEM IN RIEMANNIAN GEOMETRY

To obtain Carleman estimate at boundary, it is convenient to change variables such that, locally, the boundary is given by $x_d = 0$.

Thus we have to follow how equation change under the change of variables. To do that it is convenient to recast the problem in Riemannian geometry.

In this context, we consider $u$ as a vector field, it is a contravariant tensor and the system take the form,

$$\begin{cases} -\nabla_g r + i\mu u - bu = f, \\ -\operatorname{div}_g u + i\mu r = h. \end{cases}$$

This system is the same as above if $g = Id$.
By Riemannian geometry we follow each terms obtained by change of variables. And $g$ take the form

$$g = \begin{pmatrix} (g_{ij})_{1\le i,j\le d-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \tilde{g} & 0 \\ 0 & 1 \end{pmatrix}$$

# CONJUGATED SYSTEM

From

$$\begin{cases} -\nabla_g r + i\mu u = f \text{ in } x_d > 0, \\ -\operatorname{div}_g u + i\mu r = h \text{ in } x_d > 0, \\ u^d = 0 \text{ on } x_d = 0, \end{cases}$$

let $v = e^{\tau\varphi} u$ and $w = e^{\tau\varphi} r$. We have

$$\nabla_g r = e^{-\tau\varphi}\big(\nabla_g w - \tau w \nabla_g \varphi\big),$$
$$\operatorname{div}_g u = e^{-\tau\varphi}\big(\operatorname{div}_g v - \tau g(\nabla_g \varphi, v)\big).$$

$$\begin{cases} -i\nabla_{\tilde{g}} w + i\tau w \nabla_{\tilde{g}} \varphi - \mu \tilde{v} = i\tilde{F} \text{ in } x_d > 0, \\ -i\partial_{x_d} w + i\tau w \partial_{x_d} \varphi - \mu v^d = iF^d \text{ in } x_d > 0, \\ -i \operatorname{div}_g v + i\tau \tilde{g}(\nabla_{\tilde{g}} \varphi, \tilde{v}) + i\tau v^d \partial_{x_d} \varphi - \mu w = iG \text{ in } x_d > 0, \\ v^d = 0 \text{ on } x_d = 0. \end{cases}$$

We have $\operatorname{div}_g v = \partial_{x_d} v^d + \operatorname{div}_{\tilde{g}} \tilde{v}$.

# REDUCED SYSTEM

Here we consider the system on $v^d$ and $w$.

$$\begin{cases} D_{x_d} w + i\tau w \partial_{x_d}\varphi - \mu v^d = iF^d \text{ in } x_d > 0, \\ D_{x_d} v^d + \mu^{-1}\mathrm{Op}_{\mathsf{T}}(\delta)\mathrm{Op}_{\mathsf{T}}(\zeta')w - \mu w + i\tau v^d \partial_{x_d}\varphi - ihv^d = \tilde{G} \\ v^d = 0 \text{ on } x_d = 0, \end{cases}$$

where $\tilde{G} = iG + i\mu^{-1}\mathrm{Op}_{\mathsf{T}}(\delta)\tilde{F}$,
$\mathrm{Op}_{\mathsf{T}}(\delta) := -i\,\mathrm{div}_{\tilde{g}} + i\tau\tilde{g}(\nabla_{\tilde{g}}\varphi, \cdot)$,
$\mathrm{Op}_{\mathsf{T}}(\zeta') = -i\nabla_{\tilde{g}} + i\tau\nabla_{\tilde{g}}\varphi$,
Let $U = (w, v^d)$, the system has the form

$$D_{x_d} U - BU = H, \text{ where } H = (iF^d, iG + \mu^{-1}\mathrm{Op}_{\mathsf{T}}(\delta)\tilde{F}),$$

and $B$ is a tangential matrix operators with principal symbol

$$b = \begin{pmatrix} -i\tau\partial_{x_d}\varphi & \mu \\ -\mu^{-1}q(x,\xi') + \mu & -i\tau\partial_{x_d}\varphi \end{pmatrix},$$

$$q(x,\xi') = \sum_{1\le i,j\le d-1} g^{ij}(x)\big(\xi_i + i\tau\partial_{x_i}\varphi(x)\big)\big(\xi_j + i\tau\partial_{x_j}\varphi(x)\big).$$

# IDEAS TO ESTIMATES $U$

Ideas to obtain estimates on $U$, solution of $D_{x_d}U - BU = H$.
Assume we have a left eigenvector $\ell$ to $b$, i.e. $\ell b = \lambda\ell$.
We compose the system by $\mathrm{Op}_{\mathsf{T}}\ell$, modulo error terms we have

$$D_{x_d}(\mathrm{Op}_{\mathsf{T}}(\ell)U) - \mathrm{Op}_{\mathsf{T}}(\lambda)(\mathrm{Op}_{\mathsf{T}}(\ell)U) = \mathrm{Op}_{\mathsf{T}}(\ell)H.$$

We have three cases.

1) $\mathrm{Im}\,\lambda < 0$, one obtains an elliptic interior estimate and one estimates the trace by $H$
2) $\mathrm{Im}\,\lambda > 0$, one obtains an elliptic interior estimate by the trace and $H$
3) $\mathrm{Im}\,\lambda = 0$, one obtains an interior estimate with $1/2$ lost of derivative by the trace and $H$, typic for Carleman estimates.

To do that we have some constraints, $\lambda$ and $\ell$ should be smooth, for instance $\lambda$ simple. We need enough left eigenvectors to estimate $U$.

# ALGEBRAIC RESULTS ON $b$

Characteristic polynomial of

$$b = \begin{pmatrix} -i\tau\partial_{x_d}\varphi & \mu \\ -\mu^{-1}q(x,\xi') + \mu & -i\tau\partial_{x_d}\varphi \end{pmatrix},$$

is given by $P(\lambda) = (\lambda + i\tau\partial_{x_d}\varphi)^2 + q - \mu^2$.

Let $\alpha$ be such that $\alpha^2 = q - \mu^2$ with $\operatorname{Re}\alpha \geq 0$.

The roots of $P$ are $-i\tau\partial_{x_d}\varphi \pm i\alpha$.

If $\alpha = 0$ the previous analysis does not work and we have to treat this case by an other method.

If $\alpha \neq 0$, roots are simple and we have to study sign of the imaginary part of roots. One has

$$\operatorname{Im}(-i\tau\partial_{x_d}\varphi \pm i\alpha) = -\tau\partial_{x_d}\varphi \pm \operatorname{Re}\alpha$$

# SIGN OF IMAGINARY PART OF ROOTS

Now we assume $\partial_{x_d}\varphi > 0$ at the boundary. From

$$\operatorname{Im}(-i\tau\partial_{x_d}\varphi \pm i\alpha) = -\tau\partial_{x_d}\varphi \pm \operatorname{Re}\alpha$$

1) If $|\operatorname{Re}\alpha| < |\tau\partial_{x_d}\varphi|$, both roots have negative imaginary parts.
2) If $|\operatorname{Re}\alpha| > |\tau\partial_{x_d}\varphi|$, one root has a negative imaginary part and the other a positive imaginary part.
3) $|\operatorname{Re}\alpha| = |\tau\partial_{x_d}\varphi|$, one root is real and the other has a negative imaginary part.

To obtain a convenient condition we apply the following result.

Denoting $s = t^2$ where $t, s \in \mathbb{C}$ we have for $r_0 > 0$,

$$|\operatorname{Re} t| \substack{<\\=\\>} r_0 \iff 4r_0^2 \operatorname{Re} s - 4r_0^4 + (\operatorname{Im} s)^2 \substack{<\\=\\>} 0.$$

Taking $r_0 = |\tau\partial_{x_d}\varphi|$ and $s = q - \mu^2$, we obtain a condition on $q$.

# DIAGONALIZED SYSTEM

Summary, the symbolic matrix is

$$b = \begin{pmatrix} -i\tau\partial_{x_d}\varphi & \mu \\ -\mu^{-1}q(x,\xi') + \mu & -i\tau\partial_{x_d}\varphi \end{pmatrix},$$

The eigenvalues are $-i\tau\partial_{x_d}\varphi \pm i\alpha$,

The associated eigenvectors are $\begin{pmatrix} \mp i\alpha & \mu \end{pmatrix}$.

We have always $\mathrm{Im}(-i\tau\partial_{x_d}\varphi - i\alpha) < 0$ (remember $\mathrm{Re}\,\alpha \geq 0$.)

We define (in fact we need some microlocal cut-off)

$$z_j = (-1)^j i\Lambda_{\mathsf{T},\tau}^{-1}\mathrm{Op}_{\mathsf{T}}(\alpha)w + \mu\Lambda_{\mathsf{T},\tau}^{-1}v^d,$$

where $\Lambda_{\mathsf{T},\tau}^{-1} = \mathrm{Op}_{\mathsf{T}}((\tau^2 + |\xi'|^2)^{-2})$.

On boundary $x_d = 0$, we have $z_1 + z_2 = 0$ as $v^d = 0$.

$$D_{x_d}z_j + \mathrm{Op}_{\mathsf{T}}\big(i\tau\partial_{x_d}\varphi + (-1)^j i\alpha\big)z_j = H_j,$$

where $H_j$ collects calculus error terms and force terms.

# ESTIMATES ON DIAGONALIZED SYSTEM

For equation

$$D_{x_d} z_j + \mathrm{Op}_{\mathsf{T}}\big(i\tau\partial_{x_d}\varphi + (-1)^j i\alpha\big) z_j = H_j,$$

if $j = 2$ as $\mathrm{Im}(-i\tau\partial_{x_d}\varphi - i\alpha) < 0$, we have an interior elliptic estimate without condition on trace. We obtain, modulo error terms

$$|\Lambda_{\mathsf{T},\tau}^{s+1/2}(z_2)_{|x_d=0}| + \|\Lambda_{\mathsf{T},\tau}^{s+1} z_2\|_+ \lesssim \|\Lambda_{\mathsf{T},\tau}^{s} H_2\|_+,$$

for all $s \in \mathbb{R}$.

For $j = 1$, the estimate depends on sign of $\mathrm{Im}(-i\tau\partial_{x_d}\varphi + i\alpha)$.
If $\mathrm{Im}(-i\tau\partial_{x_d}\varphi + i\alpha) > 0$, we have, modulo errors terms

$$\|\Lambda_{\mathsf{T},\tau}^{s+1} z_1\|_+ \lesssim \|\Lambda_{\mathsf{T},\tau}^{s} H_1\|_+ + |\Lambda_{\mathsf{T},\tau}^{s+1/2}(z_1)_{|x_d=0}|,$$

In this case both estimates yield

$$|\Lambda_{\mathsf{T},\tau}^{s+1/2}(z_2)_{|x_d=0}| + \|\Lambda_{\mathsf{T},\tau}^{s+1} z_1\|_+ + \|\Lambda_{\mathsf{T},\tau}^{s+1} z_2\|_+ \lesssim \|\Lambda_{\mathsf{T},\tau}^{s} H_1\|_+ + \|\Lambda_{\mathsf{T},\tau}^{s} H_2\|_+$$

# ESTIMATES IN CARLEMAN CASE

If $\mathrm{Im}(-i\tau\partial_{x_d}\varphi + i\alpha) = 0$, we have, modulo errors terms

$$\| \Lambda_{\mathsf{T},\tau}^{s+1/2} z_1 \|_+ \lesssim \| \Lambda_{\mathsf{T},\tau}^{s} H_1 \|_+ + |\Lambda_{\mathsf{T},\tau}^{s+1/2} (z_1)_{|x_d=0}|.$$

With the previous estimate obtained on $z_2$

$$|\Lambda_{\mathsf{T},\tau}^{s+1/2} (z_2)_{|x_d=0}| + \|\Lambda_{\mathsf{T},\tau}^{s+1} z_2\|_+ \lesssim \|\Lambda_{\mathsf{T},\tau}^{s} H_2\|_+.$$

Both estimates yield

$$|\Lambda_{\mathsf{T},\tau}^{s+1/2} (z_2)_{|x_d=0}| + \| \Lambda_{\mathsf{T},\tau}^{s+1/2} z_1 \|_+ + \|\Lambda_{\mathsf{T},\tau}^{s+1} z_2\|_+ \lesssim \|\Lambda_{\mathsf{T},\tau}^{s} H_2\|_+ + \| \Lambda_{\mathsf{T},\tau}^{s} H_1 \|_+.$$

# DOUBLE ROOTS

In cases where $-i\tau\partial_{x_d}\varphi + i\alpha = 0$ or equivalently $q - \mu^2 = 0$.

$$\begin{cases} D_{x_d} w + i\tau(\partial_{x_d}\varphi)w - \mu v^d = H_1 & \text{in } x_d > 0, \\ D_{x_d} v^d + \mu^{-1}\mathrm{Op}_{\mathsf{T}}(q - \mu^2)w + i\tau(\partial_{x_d}\varphi)v^d = H_2 & \text{in } x_d > 0, \\ v^d = 0 & \text{on } x_d = 0. \end{cases}$$

We can hope obtain first an interior elliptic estimate as both roots have negative imaginary part, and second, control of traces.

In a neighborhood of a point such that $q - \mu^2 = 0$, we have $|\xi'| \sim |\mu| \sim \tau$.

The we can hope treat the term $\mu^{-1}\mathrm{Op}_{\mathsf{T}}(q - \mu^2)w$ by perturbation as $\mu^{-1}(q - \mu^2)$ is small with respect $\tau$.

With these reductions $b$ take the form

$$b = \begin{pmatrix} i\tau\partial_{x_d}\varphi & -\mu \\ 0 & i\tau\partial_{x_d}\varphi \end{pmatrix}.$$

# ESTIMATES IN DOUBLE ROOT CASES

We obtain the model system

$$\begin{cases} D_{x_d} w + i\tau(\partial_{x_d}\varphi) w - \mu v^d = H_1 & \text{in } x_d > 0, \\ D_{x_d} v^d + i\tau(\partial_{x_d}\varphi) v^d = H_2 & \text{in } x_d > 0, \end{cases}$$

We can estimate $v^d$ by the second equation as in previous cases as $\operatorname{Im} -i\tau(\partial_{x_d}\varphi) < 0$. Then we have

$$\|\Lambda^{s+1}_{\mathsf{T},\tau} v^d\|_+ \lesssim \|\Lambda^{s}_{\mathsf{T},\tau} H_2\|_+$$

To estimate $w$ we use the first equation with the term $\mu v^d$ at the right hand side. We obtain

$$\|\Lambda^{s+1}_{\mathsf{T},\tau} w\|_+ \lesssim \|\Lambda^{s}_{\mathsf{T},\tau} H_1\|_+ + \mu \|\Lambda^{s}_{\mathsf{T},\tau} v^d\|_+.$$

This yields

$$\|\Lambda^{s+1}_{\mathsf{T},\tau} w\|_+ + \|\Lambda^{s+1}_{\mathsf{T},\tau} v^d\|_+ \lesssim \|\Lambda^{s}_{\mathsf{T},\tau} H_1\|_+ \|\Lambda^{s}_{\mathsf{T},\tau} H_2\|_+.$$

# ESTIMATES ON STATE VARIABLES

Finally the weak estimate is the one obtained by Carleman method, when one root is real.

$$|\Lambda_{\mathsf{T},\tau}^{s+1/2}(z_2)_{|x_d=0}|+\tau^{-1/2}\| \Lambda_{\mathsf{T},\tau}^{s+1} z_1\|_+ + \|\Lambda_{\mathsf{T},\tau}^{s+1} z_2\|_+ \lesssim \|\Lambda_{\mathsf{T},\tau}^{s} H_2\|_+ + \| \Lambda_{\mathsf{T},\tau}^{s} H_1\|_+.$$

As

$$\begin{aligned} z_1 &= -i\Lambda_{\mathsf{T},\tau}^{-1}\mathrm{Op}_{\mathsf{T}}(\alpha)w + \mu\Lambda_{\mathsf{T},\tau}^{-1}v^d, \\ z_2 &= i\Lambda_{\mathsf{T},\tau}^{-1}\mathrm{Op}_{\mathsf{T}}(\alpha)w + \mu\Lambda_{\mathsf{T},\tau}^{-1}v^d, \end{aligned}$$

We obtain on $w$ and $v^d$,

$$\begin{aligned} |\Lambda_{\mathsf{T},\tau}^{s+1/2}(w)_{|x_d=0}| + \tau^{-1/2}\| \Lambda_{\mathsf{T},\tau}^{s+1} w\|_+ + \tau^{1/2}\|\Lambda_{\mathsf{T},\tau}^{s} v^d\|_+ \\ \lesssim \|\Lambda_{\mathsf{T},\tau}^{s} F\|_+ + \| \Lambda_{\mathsf{T},\tau}^{s} G\|_+. \end{aligned}$$

On original variables, we have

$$\begin{aligned} \tau^{1/2}|e^{\tau\varphi}r_{|x_d=0}| + \tau^{1/2}\|e^{\tau\varphi}u\|_+ + \tau^{1/2}\|e^{\tau\varphi}r\|_+ + \tau^{-1/2}\|e^{\tau\varphi}\nabla_g r\|_+ \\ \leq C\big(\|e^{\tau\varphi}f\|_+ + \|e^{\tau\varphi}g\|_+\big). \end{aligned}$$