On a $\partial, \overline{\partial}$ system with a large parameter

Johannes Sjöstrand
IMB, Université de Bourgogne

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1. Introduction

We will discuss some asymptotic questions for the $\partial, \overline{\partial}$ problem of Dirac type, on $\mathbf{C} \simeq \mathbf{R}^2$:

$$\begin{cases} \overline{\partial}\phi_1 = \frac{q}{2} e^{\overline{kz}-kz}\phi_2, \\ \partial\phi_2 = \sigma\frac{\overline{q}}{2} e^{kz-\overline{kz}}\phi_1, \end{cases} \tag{1}$$

$$\phi_1(z) = 1 + o(1),\ \phi_2(z) = o(1),\ |z| \to \infty. \tag{2}$$

Here $k \in \mathbf{C}$ and q is a potential which is small near infinity,

$$\partial = \partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial_y}\right),\ z = x + iy.$$

$\sigma = +1$ (defocusing case) or -1 (focusing case). Our asymptotic results will be valid in both cases and from now on we take $\sigma = 1$.
Notice that $|e^{kz-\overline{kz}}| = 1$, so the exponential factors in (1) are oscillatory.

Assuming (1), (2) to have a unique solution, we have the reflection coefficient $R = R(k)$, defined by

$$\bar{R}(k) = \frac{2}{\pi} \int_{\mathbf{C}} \mathrm{e}^{kz - \bar{k}\bar{z}} \bar{q}(z) \phi_1(z; k) L(dz), \quad k \in \mathbf{C}, \tag{3}$$

where $L(dz) \simeq (1/2i)\overline{dz} \wedge dz$ is the Lebesgue measure on $\mathbf{C}$.
The map $q \mapsto R$ is called the scattering transform.
The inverse scattering transform is then given by (1) and (2) after replacing q by R and vice versa, the derivatives with respect to z by the corresponding derivatives with respect to k, and asymptotic conditions for $k \to \infty$ instead of $z \to \infty$.

The system (1), (2) appears in the study of the Davey Stuartson II equations

$$\begin{aligned} iq_t + (q_{xx} - q_{yy}) + 2(\Phi + |q|^2)q &= 0, \\ \Phi_{xx} + \Phi_{yy} + 2(|q|^2)_{xx} &= 0, \end{aligned} \tag{4}$$

and also in electrical impedance tomography.

As q in (4) evolves in time t, the reflection coefficient evolves by a trivial phase factor:

$$R(k;t) = R(k,0)e^{4it\Re(k^2)}. \tag{5}$$

The general structure, existence and uniqueness have been established by [Fo83, AbFo83, AbFo84, BeCo85, Su94a, Su94b, BrUh97, Br01, Pe16, NaReTa17]. Here we focus on the asymptotic behaviour when $|k| \to \infty$.

Numerical calculations can be carried out in a bounded region in the k-plane and it is then of interest to have asymptotic results for large k.

Plan of the talk:

- The $\overline{\partial}$ operator with polynomial weights; Carleman–Hörmander approach.
- The convergence of a perturbation series solution when $|k| \to \infty$, provided $q \in \langle \cdot \rangle^{-2} H^s(\mathbf{C})$, $1 < s \leq 2$, or $q = 1_\Omega$ where $\Omega \Subset \mathbf{C}$ is strictly convex, $\partial\Omega \in C^\infty$
- The leading asymptotics of ϕ_1, ϕ_2 when $q = 1_\Omega$ as above with $\partial\Omega$ smooth.
- Asymptotics of $R(k)$ when $q = 1_\Omega$.
- Some numerical illustrations.

2. $\overline{\partial}$ on **C** with polynomial weights.

This is very classical (Carleman–Hörmander, method of positive commutators).

Proposition

Let $\epsilon > 0$. *For every* $v \in \langle\cdot\rangle^{\epsilon-2}L^2$, *there exists* $u \in \langle\cdot\rangle^{\epsilon}L^2$ *such that*

$$\overline{\partial}u = v \text{ and } \|\langle\cdot\rangle^{-\epsilon}u\| \leq \epsilon^{-1/2}\|\langle\cdot\rangle^{2-\epsilon}v\|.$$

Proposition

When $0 < \epsilon \leq 1$ *the solution is unique and given by*

$$u(z) = \frac{1}{\pi}\int \frac{v(w)}{z-w}L(dw).$$

Roughly, forgetting about the polynomial weights, we can say that

$$(h\overline{\partial})^{-1} = \mathcal{O}(1/h) : L^2 \to L^2$$

Here $0 < h \ll 1$.
This improves to $\mathcal{O}(1)$ after a suitable microlocalization to the elliptic region, $|(\xi, \eta)| \geq 1/\mathcal{O}(1)$.

3. Application to the $\partial, \overline{\partial}$ system

Let $q \in \langle \cdot \rangle^{-2} H^s(\mathbf{C})$, $1 < s \leq 2$. Let $k \in \mathbf{C}$ with $|k| \gg 1$ and write

$$kz - \overline{kz} = \frac{i}{h} \Re(z\overline{\omega}), \;\; h = \frac{1}{|k|}, \;\; \omega = 2i \frac{\overline{k}}{|k|}.$$

Writing $\widehat{\tau}_\omega u = e^{kz - \overline{kz}} u$ (translation by ω on the h Fourier transform side), the system (1) becomes

$$\begin{cases} h\overline{\partial}\phi_1 - \widehat{\tau}_{-\omega} h \frac{q}{2} \phi_2 = 0, \\ h\partial\phi_2 - \widehat{\tau}_\omega h \frac{\overline{q}}{2} \phi_1 = 0 \end{cases} \tag{6}$$

Let $E = (h\overline{\partial})^{-1}$, $F = (h\partial)^{-1}$,

$$\mathcal{K} := \begin{pmatrix} 0 & E\widehat{\tau}_{-\omega}\frac{hq}{2}, \\ F\widehat{\tau}_{\omega}\frac{h\overline{q}}{2} & 0 \end{pmatrix} =: \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}. \tag{7}$$

Applying E and F to the two equations in (6) leads to the equivalent system

$$(1-\mathcal{K})\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{8}$$

Trying $\phi_1^0 = 1$, $\phi_2^0 = 0$, gives an error to correct. We need to solve an inhomogeneous system.

We see that $\mathcal{K} = \mathcal{O}(1) : (\langle\cdot\rangle^{\epsilon} L^2)^2 \to (\langle\cdot\rangle^{\epsilon} L^2)^2$.

However $\mathcal{K}^2 = \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix}$ is much smaller, cf. Lemma 3.2 in [Pe16]:

Proposition

$\mathcal{K}^2 = \mathcal{O}(h^{s-1}) : (\langle\cdot\rangle^{\epsilon} L^2)^2 \to (\langle\cdot\rangle^{\epsilon} L^2)^2$.

It follows that $1 - \mathcal{K}$ is bijective with inverse

$$(1-\mathcal{K}^2)^{-1}(1+\mathcal{K}) = \begin{pmatrix} (1-AB)^{-1} & 0 \\ 0 & (1-BA)^{-1} \end{pmatrix} \begin{pmatrix} 1 & A \\ B & 1 \end{pmatrix}.$$

Idea of the proof.

$$\mathcal{K}^2 = \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix},$$

$$AB = \frac{h^2}{4} E \widehat{\tau}_{-\omega} q F \widehat{\tau}_{\omega} \overline{q}.$$

Phase space localizations. Thanks to $\widehat{\tau}_{\pm\omega}$, we are always in a region where $E = \mathcal{O}(1)$ or $F = \mathcal{O}(1)$.

Proposition

When $q = 1_\Omega$ for $\Omega \Subset \mathbf{C}$ strictly convex with smooth boundary, the conclusion of the preceding proposition holds with $s = 2$.

(This is a recent improvement of the value $s = 3/2$.)

Returning to (6), we write

$$\phi_j = \phi_j^0 + \phi_j^1, \quad (\phi_1^0, \phi_2^0) = (1, 0) \tag{9}$$

and get

$$(1 - \mathcal{K}) \begin{pmatrix} \phi_1^1 \\ \phi_2^1 \end{pmatrix} = \begin{pmatrix} 0 \\ B(1) \end{pmatrix} = \mathcal{O}(1) \text{ in } \langle \cdot \rangle^\epsilon L^2,$$

leading to

Theorem

(6) has the solution (9), where

$$\phi_1^1 = (1 - AB)^{-1} AB(1), \tag{10}$$

$$\phi_2^1 = (1 - BA)^{-1} B(1). \tag{11}$$

NB: $\phi_1 = (1 - AB)^{-1}(1)$, $\phi_2 = \phi_2^1$.

4. The leading correction term when $q = 1_\Omega$.

Let $q = 1_\Omega$ be as in the last proposition and let us study the leading term in (11):

$$B(1) = F\widehat{\tau}_\omega \frac{h\overline{q}}{2}(z) = \frac{1}{2\pi}\int_\Omega \frac{1}{\overline{z}-\overline{w}} e^{kw-\overline{kw}} L(dw) =: \frac{1}{2\pi}\overline{f(z,k)}, \qquad (12)$$

NB: $A(1) = f(z,k)/(2\pi)$

$$f(z,k) = \int_\Omega \frac{1}{z-w} e^{\overline{kw}-kw} L(dw) = \iint_\Omega \frac{e^{\overline{kw}-kw}}{z-w} \frac{d\overline{w}\wedge dw}{2i}. \qquad (13)$$

Stokes' formula (integration by parts) leads to

$$f(z,k) = \frac{1}{2i\overline{k}}\int_{\partial\Omega} \frac{1}{z-w} e^{\overline{kw}-kw} dw + (\pi/\overline{k}) e^{\overline{kz}-kz} 1_\Omega(z) \qquad (14)$$

Assume first also that

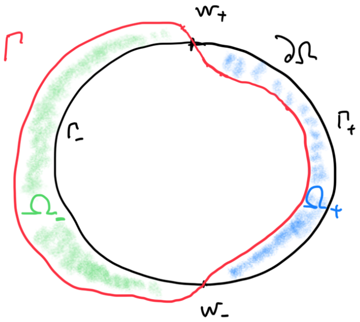
$$\partial\Omega \text{ is real analytic.} \tag{15}$$

Parametrize: $t \mapsto \gamma(t) \in \partial\Omega$, $|\dot\gamma(t)| = 1$ with the positive orientation. Let ν be the interior unit normal to $\partial\Omega$, and let

- $w_+ = w_+(k) \in \partial\Omega$ be the North pole where $\nu = c\omega$, $c < 0$,
- w_- be the South pole where $\nu = c\omega$, $c > 0$.
- Let Γ_+ be the open boundary segment from the South pole to the North pole and Γ_- the one from the North to the South.

$w_\pm$ are the critical points of $kw - \overline{kw}$ as a function on $\partial\Omega$.

Let $iu(w,\kappa)$ be a holomorphic extension of $kw - \overline{kw}$ to $\mathrm{neigh}(\partial\Omega, \mathbf{C})$. Applying the method of steepest descent, we replace $\partial\Omega$ in the integral in (14) by a contour Γ, obtained by pushing Γ_+ inwards and Γ_- outwards:



Define

$$F(z) = F_\Gamma(z) = \int_\Gamma \frac{1}{z-w} e^{-iu(w,k)} dw. \tag{16}$$

From (14) and the residue theorem, we get for $f = 2\pi A(1))$

$$f(z,k) = \frac{1}{2i\bar{k}} F(z) + (\pi/\bar{k}) \left(e^{-iu(z,k)} (1_{\Omega_-}(z) - 1_{\Omega_+}(z)) + e^{-i|k|\Re(z\overline{\omega})} 1_\Omega(z) \right). \tag{17}$$

When z is not too close to w_+ and w_-, we can apply stationary phase – steepest descent[1], to see that F is equal to

$$\begin{aligned} F(z) = \sqrt{2\pi} \Bigg(& \frac{1}{z - w_+(k)} e^{-iu(w_+(k),k) - i\pi/4} \frac{\dot{\gamma}(t_+(k))}{|\partial_t^2 u(t_+(k))|^{1/2}} \\ & + \frac{1}{z - w_-(k)} e^{-iu(w_-(k),k) + i\pi/4} \frac{\dot{\gamma}(t_-(k))}{|\partial_t^2 u(t_-(k))|^{1/2}} \Bigg) \\ & + \mathcal{O}(\langle z \rangle^{-1} k^{-3/2}). \end{aligned} \tag{18}$$

[1] making a further deformation of Γ in order to avoid the pole at $w = z$, if necessary.

When z is close to w_+ or to w_- we need to replace the corresponding term in (18) by an expression in terms of the special function

$$G(\widetilde{z}) = \int_{\widetilde{\Gamma}} \frac{1}{\widetilde{z} - \widetilde{w}} e^{-\widetilde{w}^2/2} d\widetilde{w}. \tag{19}$$

Combining this with (10), (11), leads to the following approximations for ϕ_2^1, ϕ_1^1, where $h = 1/|k|$:

$$\phi_2^1 = \frac{1}{2k} e^{i|k|\Re(\cdot\overline{\omega})} 1_\Omega + \mathcal{O}(1) h^{3/2} (\ln(1/h))^{1/2} \text{ in } \langle\cdot\rangle^\epsilon L^2, \tag{20}$$

$$\phi_1^1 = \frac{h}{4k} E(1_\Omega) + \mathcal{O}(1) h^{3/2} (\ln(1/h))^{1/2} \text{ in } \langle\cdot\rangle^\epsilon L^2. \tag{21}$$

When $\partial\Omega$ is merely smooth, this still works with $u(w,k)$ equal to an almost holomorphic extension of u_0 to $\mathrm{neigh}\,(\partial\Omega, \mathbf{C})$.

5. Asymptotics of the reflection coefficient

These results are still preliminary. Let $\mathcal{O} \Subset \mathbf{C}$ be open, strictly convex with real-analytic boundary. Let $q = 1_\Omega$ and take $\sigma = 1$ for simplicity. Let

$$D_\Omega(z) = \frac{1}{\pi} \int_\Omega \frac{1}{z - w} L(dw) \tag{22}$$

be the solution to the $\overline{\partial}$-problem:

$$\begin{cases} \partial_{\overline{z}} D_\Omega = 1_\Omega, \\ D_\Omega(z) \to 0, \ z \to \infty. \end{cases} \tag{23}$$

Example

$$D_{D(0,1)}(z) = \begin{cases} \overline{z}, \ |z| \leq 1, \\ 1/z, \ |z| \geq 1. \end{cases}$$

Theorem

D_Ω is continuous, $D_{\Omega|_\Omega} \in C^\infty(\overline{\Omega})$, $D_{\Omega|_{\mathbf{C}\setminus\overline{\Omega}}} \in C^\infty(\mathbf{C} \setminus \Omega)$. Then

$$\begin{aligned}\overline{R} =& \frac{2}{\pi}\int_\Omega e^{kz-\overline{kz}} L(dz) \\ &+ \frac{1}{4i\pi|k|^2}\left(-\int_{\widetilde{\Gamma}} D_\Omega(w)e^{iu(w,k)}dw + \overline{\int_\Gamma D_\Omega(w)e^{-iu(w,k)}dw}\right) \\ &+ \mathcal{O}(|k|^{-3}\ln|k|),\end{aligned} \tag{24}$$

When $\Omega = D(0,1)$ is the unit disc, numerical computations indicate that

$$R \approx R_{\text{asym}} := \frac{1}{\sqrt{\pi k^3}}\left(\sin(2k-\pi/4) - \frac{5}{16k}\cos(2k-\pi/4)\right),\ k \to +\infty,. \tag{25}$$

and that $R - R_{\text{asym}} = \mathcal{O}(|k|^{-7/2})$.

Need to study AB where A, B are given in (7),

$$Au(z) = \frac{1}{2\pi} \int_\Omega \frac{1}{z - w} e^{-kw + \overline{kw}} u(w) L(\mathrm{d}w),$$

$$Bu(z) = \frac{1}{2\pi} \int_\Omega \frac{1}{\overline{z} - \overline{w}} e^{kw - \overline{kw}} u(w) L(\mathrm{d}w),$$

$$ABu(z) = \int_\Omega K(z, w) u(w) L(\mathrm{d}w), \tag{26}$$

$$K(z, w) = \frac{1}{4\pi^2} \iint_\Omega \frac{e^{\overline{k\zeta}}}{(z - \zeta)} \frac{e^{-k\zeta}}{(\overline{\zeta} - \overline{w})} \frac{\mathrm{d}\overline{\zeta} \wedge \mathrm{d}\zeta}{2\mathrm{i}} e^{kw - \overline{kw}}, \tag{27}$$

By partitions and integration by parts, we split K into several terms that can be estimated and get,

Theorem

$$AB = \mathcal{O}(1/|k|) : \begin{cases} L^q \to L^q, & 2 < q < +\infty, \\ L^q \to \langle \cdot \rangle^{\epsilon} L^q, & 1 < q \leq 2, \ \epsilon > \frac{2}{q} - 1. \end{cases} \tag{28}$$

In particular, $AB = \mathcal{O}(1/|k|) : L^2(\Omega) \to L^2(\Omega)$.

Combining (3), (10) gives

$$\begin{aligned} \overline{R}(k) &= \frac{2}{\pi} \int_{\Omega} e^{kz - \overline{kz}} (1 - AB)^{-1}(1) L(dz) \\ &= \sum_{\nu=0}^{\infty} \frac{2}{\pi} \int_{\Omega} e^{kz - \overline{kz}} (AB)^{\nu}(1) L(dz) \end{aligned} \tag{29}$$

Define $r = r(z,k)$ by

$$A(1_\Omega) = \frac{1}{2\bar{k}} e^{-kz+\overline{kz}} 1_\Omega + r(z,k), \tag{30}$$

Comparing with (17), (16), we get

$$r = \frac{F}{4\pi i\bar{k}} - \frac{1}{2\bar{k}} e^{-iu} 1_{\Omega_+} \text{ in } \Omega. \tag{31}$$

We then get (cf (20)),

$$r(\cdot,k) = \mathcal{O}(1)|k|^{-3/2}(\ln|k|)^{1/2} \text{ in } L^2(\Omega), \tag{32}$$

$$r = \frac{F}{4\pi i\bar{k}} + \frac{\mathcal{O}(1)\ln|k|}{|k|^2} = \mathcal{O}(1)|k|^{-3/2} \text{ in } L^1(\Omega), \tag{33}$$

where $F = F(z,k)$.

Let $\langle u|v\rangle = \int uvL(dz)$ and write $A = A_k$. Then the transpose of A_k is given by $A_k^{\mathrm{t}} = -e^{-k\cdot+\overline{k\cdot}}A_{-k}e^{-k\cdot+\overline{k\cdot}}$

$$\frac{2}{\pi}\int_\Omega e^{kz-\overline{kz}}AB(1_\Omega)(z)L(dz) = \frac{2}{\pi}\langle B(1_\Omega)|A^{\mathrm{t}}e^{k\cdot-\overline{k\cdot}}(1_\Omega)\rangle$$
$$= -\frac{2}{\pi}\langle B_k(1_\Omega)|e^{-k\cdot+\overline{k\cdot}}A_{-k}(1_\Omega)\rangle.$$

Using (30) and the fact that $B_k(1_\Omega) = \overline{A_k(1_\Omega)}$, we get

$$\begin{aligned}
&\frac{2}{\pi}\int_\Omega e^{kz-\overline{kz}}AB(1_\Omega)(z)L(dz)\\
=&\frac{2}{\pi}\frac{1}{4|k|^2}\int_\Omega e^{kz-\overline{kz}}L(dz) - \frac{2}{\pi}\int_\Omega \frac{1}{2k}r(z,-k)L(dz)\\
&+\frac{2}{\pi}\int_\Omega \overline{r(z,k)}\frac{1}{2\overline{k}}L(dz) - \frac{2}{\pi}\int_\Omega e^{-kz+\overline{kz}}\overline{r(z,k)}r(z,-k)L(dz).
\end{aligned} \tag{34}$$

The 1st term in the right hand side is $\mathcal{O}(|k|^{-7/2})$. By (32) the last term is $\mathcal{O}(|k|^{-3}\ln|k|)$. Thus,

$$\begin{aligned}&\frac{2}{\pi}\int_\Omega e^{kz-\overline{kz}}AB(1_\Omega)(z)L(dz)\\&=-\frac{2}{\pi}\int_\Omega\frac{1}{2k}r(z,-k)L(dz)+\frac{2}{\pi}\int_\Omega\overline{r(z,k)}\frac{1}{2\overline{k}}L(dz)+\mathcal{O}(|k|^{-3}\ln|k|).\end{aligned} \tag{35}$$

(33) now gives

$$\frac{2}{\pi}\int_\Omega e^{kz-\overline{kz}}AB(1_\Omega)(z)L(dz)=\mathcal{O}(|k|^{-5/2}). \tag{36}$$

We shall next gain a power of k in the estimate of the general term in (29) for $\nu \geq 2$:

$$\frac{2}{\pi}\int_\Omega e^{kz-\overline{kz}}(AB)^\nu(1_\Omega)(z)L(dz) = \frac{2}{\pi}\int_\Omega e^{kz-\overline{kz}}A(BA)^{\nu-1}B(1_\Omega)(z)L(dz)$$
$$= \frac{2}{\pi}\langle (BA)^{\nu-1}B(1_\Omega)|e^{-k\cdot+\overline{k\cdot}}A_{-k}(1_\Omega)\rangle = \mathcal{O}(1)|k|^{1-\nu}|k|^{-1}|k|^{-1}$$
$$= \mathcal{O}(|k|^{-\nu-1}) = \mathcal{O}(|k|^{-3}), \quad (37)$$

since $(BA)^{\nu-1} = \mathcal{O}(|k|^{1-\nu}) : \ L^2(\Omega) \to L^2(\Omega)$ and $B(1_\Omega),\ A_{-k}(1_\Omega) = \mathcal{O}(1/|k|)$ in $L^2(\Omega)$.
Using this in (29), we get

$$\overline{R} = \frac{2}{\pi}\int_\Omega e^{kz-\overline{kz}}L(dz) + \frac{2}{\pi}\int_\Omega e^{kz-\overline{kz}}AB(1_\Omega)L(dz) + \mathcal{O}(|k|^{-3}), \quad (38)$$

and in particular,

$$\overline{R} = \frac{2}{\pi}\int_\Omega e^{kz-\overline{kz}}L(dz) + \mathcal{O}(|k|^{-5/2}). \quad (39)$$

We next study the second term in the right hand side of (38), starting from (35). Using (32), (33) in (35), we get

$$\frac{2}{\pi}\int_\Omega e^{kz-\overline{kz}}AB(1_\Omega)(z)L(dz) =$$
$$\frac{2}{\pi}\int_\Omega \frac{1}{2k}\frac{F(z,-k)}{4i\pi\overline{k}}L(dz) + \frac{2}{\pi}\int_\Omega \frac{\overline{F}(z,k)}{-4i\pi k}\frac{1}{2\overline{k}}L(dz) + \mathcal{O}(|k|^{-3}\ln|k|)$$
$$= \frac{1}{4i\pi^2|k|^2}\int_\Omega (F(z,-k) - \overline{F}(z,k))L(dz) + \mathcal{O}(|k|^{-3}\ln|k|). \quad (40)$$

Here

$$\int_\Omega F(z,-k)L(dz) = \int_\Omega\int_{\widetilde{\Gamma}} \frac{1}{z-w}e^{iu(w,k)}dwL(dz)$$
$$= \int_{\widetilde{\Gamma}}\int_\Omega \frac{1}{z-w}L(dz)e^{iu(w,k)}dw = -\pi\int_{\widetilde{\Gamma}} D_\Omega(w)e^{iu(w,k)}dw, \quad (41)$$

using that $\frac{1}{z-w}e^{iu(w,k)}$ is integrable on $\Omega\times\widetilde{\Gamma}$ for the measure $L(dz)|dw|$.

Similarly,

$$-\int_{\Omega}\overline{F(z,k)}L(dz)=\overline{\int_{\Omega}\int_{\Gamma}\frac{1}{w-z}e^{-iu(w,k)}dwL(dz)}$$
$$=\overline{\int_{\Gamma}\int_{\Omega}\frac{1}{w-z}L(dz)e^{-iu(w,k)}dw}=\pi\overline{\int_{\Gamma}D_{\Omega}(w)e^{-iu(w,k)}dw}. \quad (42)$$

Using (41), (42) in (40), we get

$$\frac{2}{\pi}\int_{\Omega}e^{kz-\overline{kz}}AB(1_{\Omega})(z)L(dz)=$$
$$\frac{1}{4i\pi|k|^2}\left(-\int_{\widetilde{\Gamma}}D_{\Omega}(w)e^{iu(w,k)}dw+\overline{\int_{\Gamma}D_{\Omega}(w)e^{-iu(w,k)}dw}\right)+\mathcal{O}(|k|^{-3}\ln|k|) \quad (43)$$

Using this in (38) gives (24).

6. Numerics

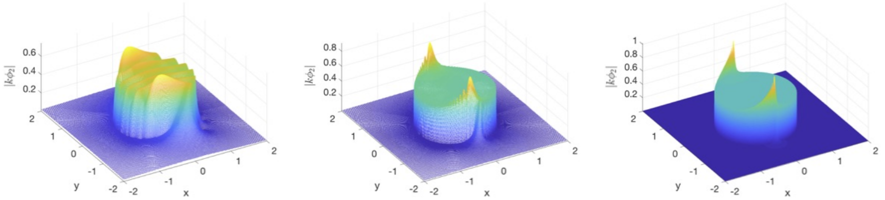


Figure: The solution ϕ_2 for the characteristic function of the disk multiplied by k for $k = 10, 100, 1000$ from left to right.

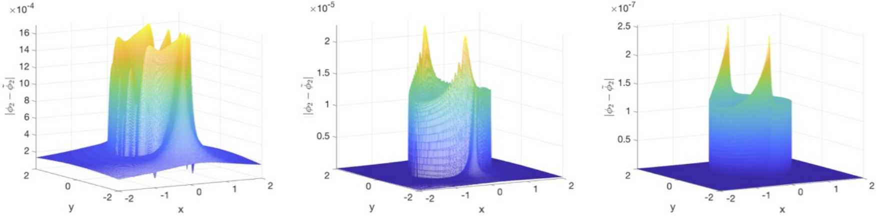


Figure: Difference between the solution ϕ_2 for the characteristic function of the disk and $\overline{f}/(2\pi)$ for $k = 10, 100, 1000$ from left to right.

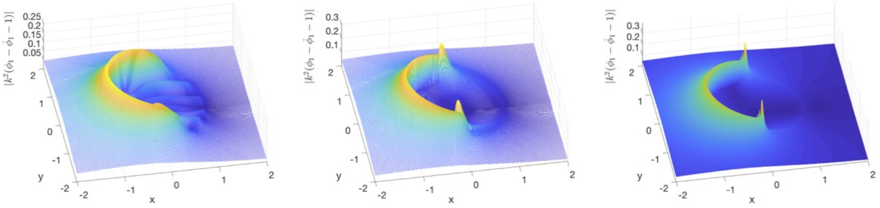


Figure: Difference between the solution ϕ_1 for the characteristic function of the disk and $1 + \frac{\bar{z}}{4k}$ multiplied by k^2 for $k = 10, 100, 1000$ from left to right.

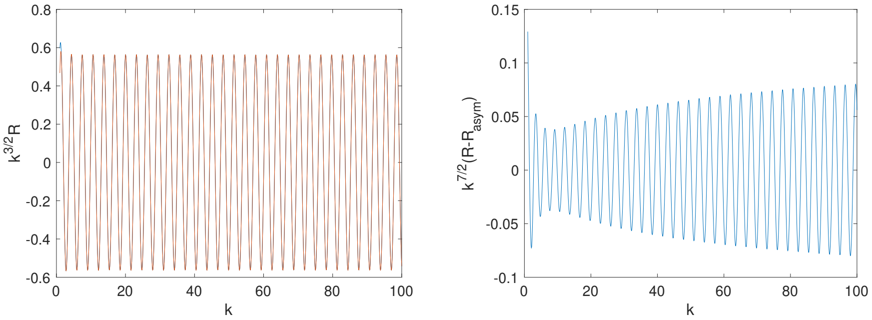


Figure: Reflection coefficient for the characteristic function of the disk, on the left R in blue and R_{asym} from (25) in red, both multiplied with $k^{3/2}$, on the right the difference between both multiplied with $k^{7/2}$.

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